There may be exactly 27 Q-points

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Introduction

- $[\omega]^\omega$ denotes the set of infinite subsets of ω.
- $\mathcal{F} \subseteq [\omega]^\omega$ is a *filter* if $\mathcal{F}$ is closed under finite intersections and under supersets.
- $\mathcal{U} \subseteq [\omega]^\omega$ is an *ultrafilter* if $\mathcal{U}$ is a filter and for all $x \subseteq \omega$: either $x \in \mathcal{U}$ or $\omega \setminus x \in \mathcal{U}$.
- A subset $\mathcal{B}$ of a filter $\mathcal{F}$ is a *basis* of $\mathcal{F}$ if $\mathcal{F} = \{y \in [\omega]^\omega : \exists x \in \mathcal{B} : y \supseteq x\}$. If $\mathcal{B}$ is a basis of $\mathcal{F}$ and $|\mathcal{B}| = \kappa$, we say that $\mathcal{F}$ is κ-*generated*.

Introduction

- An ultrafilter $\mathcal{U}$ is a **Ramsey ultrafilter** if for every partition $\{P_n : n \in \omega\}$ of ω, there either exists some $n \in \omega$ such that $P_n \in \mathcal{U}$, or there exists some $x \in \mathcal{U}$ such that $\forall n \in \omega : |x \cap P_n| \leq 1$.
- An ultrafilter $\mathcal{U}$ is a **P-point** if for every partition $\{P_n : n \in \omega\}$ of ω, there either exists some $n \in \omega$ such that $P_n \in \mathcal{U}$, or there exists some $x \in \mathcal{U}$ such that $\forall n \in \omega : x \cap P_n$ *is finite*.
- An ultrafilter $\mathcal{U}$ is a **Q-point** if for every partition $\{P_n : n \in \omega\}$ of ω *into finite sets*, there exists some $x \in \mathcal{U}$ such that $\forall n \in \omega : |x \cap P_n| \leq 1$.

Introduction

$\mathcal{U}$ is a Ramsey ultrafilter if and only if it is both a P-point and a Q-point.

The main theorem

Main Theorem (Halbeisen, H., Özalp [3])

For any $n \in \omega$, it is consistent with ZFC that there exist exactly n Q-points.

There may be many Q-points/P-points/Ramsey ultrafilters

Consider the following cardinal characteristics of the continuum:

- The *continuum* $\mathfrak{c} := |\mathbb{R}| = 2^\omega$
- The *dominating number*

$$\mathfrak{d} := \min\left\{ |\mathcal{F}| : \begin{array}{l} \mathcal{F} \subseteq {}^\omega\omega \text{ is such that} \\ \forall g \in {}^\omega\omega\ \exists f \in \mathcal{F}\ \forall n \in \omega : g(n) < f(n). \end{array} \right\}$$

- The *covering number of the meager ideal*

$$\mathrm{cov}(\mathcal{M}) := \min\left\{ |\mathcal{F}| : \begin{array}{c} \mathcal{F} \text{ is a family of nowhere dense} \\ \text{subsets of } \mathbb{R} \text{ such that } \bigcup \mathcal{F} = \mathbb{R}. \end{array} \right\}.$$

$$\mathrm{ZFC} \vdash \omega_1 \leq \mathrm{cov}(\mathcal{M}) \leq \mathfrak{d} \leq \mathfrak{c}$$

There may be many Q-points/P-points/Ramsey ultrafilters

$$\text{ZFC} \vdash \omega_1 \leq \text{cov}(\mathcal{M}) \leq \mathfrak{d} \leq \mathfrak{c}$$

- If $\text{cov}(\mathcal{M}) = \mathfrak{d}$, then Q-points exist.
- If $\mathfrak{d} = \mathfrak{c}$, then P-points exist.
- If $\text{cov}(\mathcal{M}) = \mathfrak{c}$, then Ramsey ultrafilters exist.

In fact:

- $\text{cov}(\mathcal{M}) = \mathfrak{d} \iff$ every $< \mathfrak{d}$-generated filter can be extended to $2^{\mathfrak{c}}$ distinct Q-points (Millán [6]).
- $\mathfrak{d} = \mathfrak{c}$ and no ultrafilter is $< \mathfrak{c}$-generated $\iff$ every $< \mathfrak{c}$-generated filter can be extended to $2^{\mathfrak{c}}$ distinct P-points (Millán [7]).
- $\text{cov}(\mathcal{M}) = \mathfrak{c} \iff$ every $< \mathfrak{c}$-generated filter can be extended to $2^{\mathfrak{c}}$ distinct Ramsey ultrafilters (Millán [7]).

There may be no Q-points/P-points/Ramsey ultrafilters

However, the existence of Q-points, P-points and Ramsey ultrafilters cannot be proven in ZFC alone:

- Kunen [4], 1975: It is consistent that there are no Ramsey ultrafilters.
- Shelah [10], 1977: It is consistent that there are no P-points.
- Miller [8], 1978: It is consistent that there are no Q-points.

  (They do not exist in the Mathias- and in the Laver model.)

Open problem:

Is it consistent that there are no P-points and no Q-points simultaneously?

There may be only few P-points/Ramsey ultrafilters

- Note: We care about the total number of P-points/Q-points/Ramsey ultrafilters *up to isomorphism*, i.e., up to permutations of ω.
- Note: If there are $2^{\mathfrak{c}}$ distinct ultrafilters of a given type, then there are $2^{\mathfrak{c}}$ ultrafilters of this type up to isomorphism.

There may be exactly 27 P-points/Ramsey ultrafilters

Shelah [9]: For any cardinal $0 \leq \kappa \leq \omega_2$, there is a model of ZFC in which $\mathfrak{c} = \omega_2$ and in which there exist exactly κ P-points up to isomorphism (and all of them are Ramsey ultrafilters).

How to have only few Q-points

- Say that two ultrafilters $\mathcal{U}$ and $\mathcal{V}$ are *nearly coherent* if there exists some finite-to-one $f : \omega \to \omega$ such that $f(\mathcal{U}) = f(\mathcal{V})$.
  (where $f(\mathcal{U}) := \{x \subseteq \omega : f^{-1}[x] \in \mathcal{U}\}$)
- This is an equivalence relation, call each equivalence class a *near-coherence class*.
- Note: If $\mathcal{U}$ is a Q-point and f is finite-to-one, then $\mathcal{U}$ is isomorphic to $f(\mathcal{U})$. Hence, for Q-points $\mathcal{U}$ and $\mathcal{V}$,

  $\mathcal{U}$ and $\mathcal{V}$ are nearly coherent $\iff$ $\mathcal{U}$ and $\mathcal{V}$ are isomorphic.

- Therefore: The number of Q-points is at most the number of near-coherence classes.

How to have only few Q-points

- Banakh, Blass [1]: The number of near-coherence classes is either finite or $2^{\mathfrak{c}}$. Moreover, if there is no $< \mathfrak{d}$-generated ultrafilter, then the number of near-coherence classes is $2^{\mathfrak{c}}$.
- Note: If $\mathcal{U}$ is κ-generated and $f : \omega \to \omega$, then $f(\mathcal{U})$ is also κ-generated. Hence: If a Q-point $\mathcal{V}$ is nearly-coherent to a κ-generated ultrafilter $\mathcal{U}$, then $\mathcal{V}$ is κ-generated.
- Note: A Q-point cannot be $< \mathfrak{d}$-generated.

  (Why? If $\mathcal{B}$ is a basis of a Q-point and $f_x \in {}^{\omega}\omega$ enumerates $x \in \mathcal{B}$, then $\{f_x : x \in \mathcal{B}\}$ is a dominating family.)

$\implies$ If the number of near-coherence classes is finite, then the number of Q-points is at most the number of near-coherence classes **minus one**.

How to have only few Q-points

- Blass, Shelah [2]: There is a model with only one near-coherence class (and hence no Q-point).
- Mildenberger [5]: There are models with exactly two and with exactly three near-coherence classes.
  These models contain exactly one and exactly two Q-points, respectively.

Open problem: Are exactly *n* near-coherence classes consistent for $4 \leq n < \omega$?

A different approach

Main Theorem.

It is consistent there exist exactly 27 Q-points.

Sketch of proof.

Let $\bar{\mathcal{U}} := \langle \mathcal{U}_0, \mathcal{U}_1, ..., \mathcal{U}_{26} \rangle$ be a sequence of 27 pairwise non-isomorphic Ramsey ultrafilters. Consider the forcing notion $\mathbb{M}_{\bar{\mathcal{U}}} = \langle M, \leq \rangle$, where

$$M = \{\langle s, X_0, X_1, ..., X_{26} \rangle : s \in \text{fin}(\omega),\ \forall i < 27 : X_i \in \mathcal{U}_i \text{ and } \max s < \min X_i\}$$

and

$$\langle s, X_0, ..., X_{26} \rangle \leq \langle t, Y_0, ..., Y_{26} \rangle :\iff \begin{cases} s \supseteq t \text{ and} \\ \forall i < 27 : X_i \subseteq Y_i \text{ and} \\ \text{if } m \in s \setminus t \text{ is the i'th element of} \\ s \bmod 27, \text{then } m \in Y_i. \end{cases}$$

There may be exactly 27 Q-points

What does forcing with $\mathbb{M}_{\bar{\mathcal{U}}}$ do?

- adds an infinite pseudo-intersection η_i of each $\mathcal{U}_i$.

  ($\forall i < 27\ \forall X \in \mathcal{U}_i\ \exists m \in \omega : \eta_i \setminus m \subseteq X$)
- satisfies the following **Lemma**:
  Assume $\mathcal{V}$ is a Q-point not isomorphic to any of the $\mathcal{U}_i$. Forcing first with $\mathbb{M}_{\bar{\mathcal{U}}}$ and then with a forcing notion satisfying the *Laver property*, we obtain a model in which no extension of $\mathcal{V}$ to an ultrafilter is a Q-point.
- $\mathbb{M}_{\bar{\mathcal{U}}}$ satisfies the Laver property.

  (Hence, countable support iterations of $\mathbb{M}_{\bar{\mathcal{U}}}$ satisfy the Laver property.)

There may be exactly 27 Q-points

Construction:

- Start in a model with the Continuum Hypothesis, let $\bar{\mathcal{U}} = \langle \mathcal{U}_0, \mathcal{U}_1, ..., \mathcal{U}_{26} \rangle$ be a sequence of 27 pairwise non-isomorphic Ramsey ultrafilters.
- Force with $\mathbb{M}_{\bar{\mathcal{U}}}$.
- Extend the $\mathcal{U}_i \cup \{\eta_i\}$ to non-isomorphic Ramsey ultrafilters $\mathcal{U}_i'$.
- Force with $\mathbb{M}_{\bar{\mathcal{U}}'}$.
- Extend the $\mathcal{U}_i' \cup \{\eta_i'\}$ to non-isomorphic Ramsey ultrafilters $\mathcal{U}_i''$.
- Force with $\mathbb{M}_{\bar{\mathcal{U}}''}$.
- ...

After doing this ω_2-many times (with countable supports), we end up with 27 Ramsey ultrafilters, and by the **Lemma**, these will be the only Q-points. $\square$

There may be exactly 27 Q-points

- Note: The forcing notion $\mathbb{M}_{\bar{\mathcal{U}}}$ can be seen as an ($n = 27$)-dimensional variant of *relativized Mathias forcing*. If we consider classical Mathias forcing to be the 0-dimensional variant, we obtain the Mathias model, which indeed contains 0 Q-points by the old Miller result [8].
- Note: No ultrafilter in the above model is $< \mathfrak{d}$-generated, hence it contains $2^{\mathfrak{c}}$ near-coherence classes.
- **Open Problem:** Is it consistent that there are exactly $\aleph_0$ Q-points?

  Complication: If there are infinitely many Ramsey ultrafilters, there are $2^{\mathfrak{c}}$ Q-points.

Thank you for your attention!

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