Laver ultrafilters

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Introduction

Definition

Recall that a forcing notion $\mathbb{P}$ has the *Laver property* if the following holds for every $\mathbb{P}$-generic extension $\mathbf{V}[G]$:

If $f \in \omega^\omega \cap \mathbf{V}[G]$ and there exists $g \in \omega^\omega \cap \mathbf{V}$ with $f < g$, then there exists $S \in (\prod_{n\in\omega}[g(n)]^{\leq n+1}) \cap \mathbf{V}$ with $f(n) \in S(n)$ for each $n \in \omega$.

Note:

- If $\mathbb{P}$ has the Laver property, then $\mathbb{P}$ does not add Cohen reals.
- The Laver property is preserved under countable support iterations of proper forcing notions.

Definition

Let $\mathcal{U}$ be an ultrafilter over ω. The forcing notion $\mathbb{L}_{\mathcal{U}}$ (Laver forcing relativized to $\mathcal{U}$) consists of trees $T \subseteq \omega^{<\omega}$ such that for each $\mathrm{stem}(T) \subseteq s \in T : \mathrm{succ}_T(s) := \{n \in \omega : s^\frown n \in T\} \in \mathcal{U}$, ordered by inclusion.

Question: For what ultrafilters $\mathcal{U}$ does $\mathbb{L}_{\mathcal{U}}$ have the Laver property?

Call such a $\mathcal{U}$ a *Laver ultrafilter*.

Laver ultrafilters

Theorem

$\mathcal{U}$ is a Laver ultrafilter if and only if the following holds:

For every sequence $\langle \mathcal{P}_n : n \in \omega \rangle$ of partitions of ω into finitely many sets, there exists $A \in \mathcal{U}$ such that for every $n \in \omega$, A has non-empty intersection with at most $n+1$ elements of $\mathcal{P}_n$.

Some easy corollaries:

(i) Every rapid P-point is a Laver ultrafilter.

(ii) Every Laver ultrafilter is rapid.

(iii) If $\mathcal{U}$ is a Laver ultrafilter and $\mathcal{V} \leq_{\mathrm{RK}} \mathcal{U}$, then $\mathcal{V}$ is a Laver ultrafilter.

(iv) If $\mathcal{U}, \mathcal{V}_i, i \in \omega$, are Laver ultrafilters, then $\mathcal{U} - \sum_{i \in \omega} \mathcal{V}_i$ is a Laver ultrafilter.

where $\mathcal{U} - \sum_{i \in \omega} \mathcal{V}_i$ is the ultrafilter on $\omega \times \omega$ consisting of those $A \subseteq \omega \times \omega$ satisfying $\{i \in \omega : \{j \in \omega : \langle i, j \rangle \in A\} \in \mathcal{V}_i\} \in \mathcal{U}$.

Laver ultrafilters in the Baumgartner framework

Definition (Baumgartner [1])

Let $\mathcal{I}$ be a collection of subsets of some set X (in our case, $X = 2^\omega$ or $X = \mathbb{Q} \subseteq 2^\omega$) such that $\mathcal{I}$ contains all singletons and is closed under subsets.

Call an ultrafilter $\mathcal{U}$ over ω an *$\mathcal{I}$-ultrafilter* if for every $F : \omega \to X$, there exists some $A \in \mathcal{U}$ with $F[A] \in \mathcal{I}$.

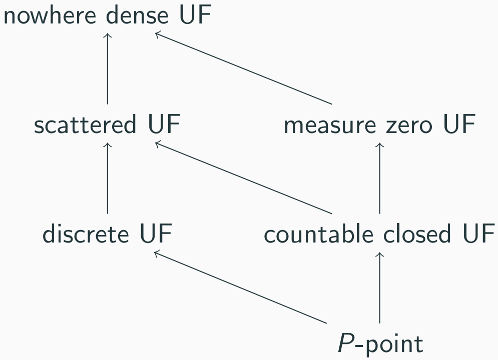
Laver ultrafilters in the Baumgartner framework

Some $\mathcal{I}$ of interest:

- the nowhere dense subsets of 2^ω. Call $\mathcal{I}$-ultrafilters *nowhere dense ultrafilters*.
- the subsets of 2^ω with closure of measure zero, call $\mathcal{I}$-ultrafilters *measure zero ultrafilters*.
- the scattered subsets of 2^ω, call $\mathcal{I}$-ultrafilters *scattered ultrafilters*.
- the subsets of 2^ω with countable closure, call $\mathcal{I}$-ultrafilters *countable closed ultrafilters*.
- the discrete subsets of 2^ω, call $\mathcal{I}$-ultrafilters *discrete ultrafilters*.

Laver ultrafilters in the Baumgartner framework

Containments among these ultrafilter classes provable in ZFC (Brendle [3]).



An arrow ($\mathcal{I}$-ultrafilter) $\rightarrow$ ($\mathcal{J}$-ultrafilter) means that every $\mathcal{I}$-ultrafilter is a $\mathcal{J}$-ultrafilter.

Definition

For $A \subseteq 2^\omega$ and $n \in \omega$, define $\mathrm{level}_A(n) := |\{x|_n : x \in A\}|$.

For a non-decreasing, unbounded function $f \in \omega^\omega$ consider the collection

$$\mathcal{I}_f := \{A \subseteq 2^\omega : \mathrm{level}_A \leq f + 1\}.$$

Lemma

$\mathcal{U}$ is a Laver ultrafilter if and only if $\mathcal{U}$ is an $\mathcal{I}_f$-ultrafilter for each $f \in \omega^\omega$ as above.

Laver ultrafilters in the Baumgartner framework

Observe: Each $\mathcal{I}_f$ consists of measure zero sets, and $A \in \mathcal{I}_f \iff \bar{A} \in \mathcal{I}_f$, hence every Laver ultrafilter is a measure zero ultrafilter.

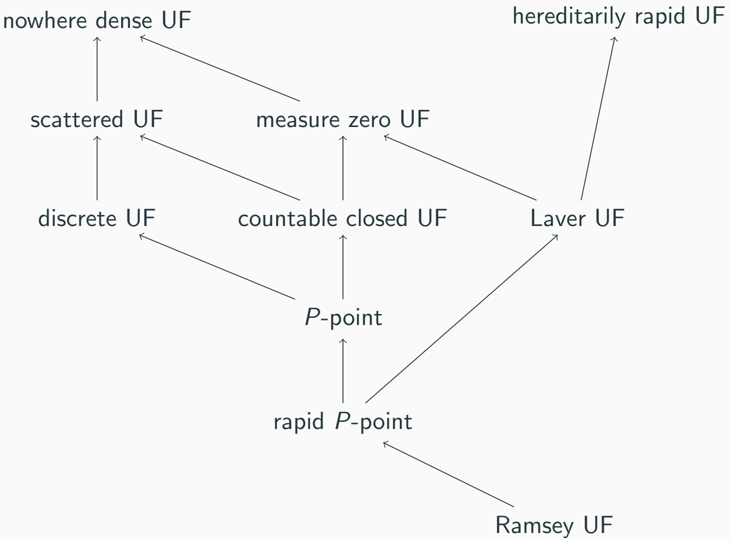
Compare:

Theorem (Błaszczyk, Shelah [2])

$\mathbb{L}_{\mathcal{U}}$ does not add Cohen reals if and only if $\mathcal{U}$ is a nowhere dense ultrafilter.

Side note: Laver ultrafilters are $\mathcal{Y}^0_f$-ultrafilters for every strictly increasing $f \in \omega^\omega$, where $\mathcal{Y}^0_f$ denotes the ideal of subsets of 2^ω with closure in the *Yorioka ideal* $\mathcal{Y}_f$ (approximations of the *strong measure zero* ideal $\mathcal{SN}$, [5]).

Extended map of ZFC-implications



This map is complete:

Theorem (Brendle, Flašková [4])

MA(countable) implies the existence of a hereditarily rapid ultrafilter that is not nowhere dense.

Theorem

MA($\sigma -$ linked) implies the existence of a Laver ultrafilter that is not scattered.

Open problem: Does MA imply the existence of an ultrafilter that is both countable closed and hereditarily rapid, but not a Laver ultrafilter?

(Generic) existence of Laver ultrafilters

Definition

An ultrafilter class $\mathcal{C}$ *exists generically* if every filter base of cardinality $< \mathfrak{c}$ can be extended to an ultrafilter in the class $\mathcal{C}$.

Theorem

(i) If $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ or $\mathrm{non}(\mathcal{NA}) = \mathfrak{c}$, then Laver ultrafilters exist generically.

(ii) If Laver ultrafilters exist generically, then $\mathrm{non}(\mathcal{SN}) = \mathfrak{c}$ and $\max\{\mathrm{non}(\mathcal{E}), \mathfrak{d}\} = \mathfrak{c}$.

Open problem: Is the generic existence of Laver ultrafilters equivalent to $\max\{\mathrm{cov}(\mathcal{M}), \mathrm{non}(\mathcal{NA})\} = \mathfrak{c}$?

(Generic) existence of Laver ultrafilters

Using (i):

Theorem

It is consistent that there are no P-points while Laver ultrafilters exist generically.

Proof.

Kill P-points using Grigorieff forcing and simultaneously increase non($\mathcal{NA}$). $\square$

Thank you for your attention!

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