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Global Tower Spectra and Other Generalized Cardinal Characteristics

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Abstract

We prove old and new results on cardinal characteristics of the generalized Baire space ${}^{\kappa}\kappa$. The first part of this thesis consists of a detailed exposition of important classical results on the characteristics $\mathfrak{s}(\kappa)$ and $\mathfrak{b}(\kappa)$ – the generalized splitting and bounding numbers, respectively. More concretely, we prove that the characteristics $\mathfrak{s}(\omega)$ and $\mathfrak{b}(\omega)$ are independent and that, in stark contrast, $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$ is provable in ZFC for uncountable κ. We further show how the value of $\mathfrak{s}(\kappa)$ is linked to two large cardinal assumptions on κ.

In the second part, we prove new results for $\mathfrak{s}(\kappa)$ and $\mathfrak{b}(\kappa)$ and, in particular, for the characteristic $\mathfrak{sp}(\mathfrak{t}(\kappa))$ – the generalized tower spectrum. These new results are mainly concerned with controlling cardinal characteristics globally, i.e., at many regular cardinals κ simultaneously. We first show how to force $\mathfrak{s}(\kappa) < \mathfrak{b}(\kappa)$ at many strongly unfoldable cardinals κ simultaneously, based on previous work by Bağ and Fischer. In the main chapter, we prove that both small and large generalized tower spectra at all regular κ simultaneously are consistent and that globally, a small tower spectrum is consistent with an arbitrarily large spectrum of maximal almost disjoint families. Finally, we show the consistency of any non-trivial upper bound on $\mathfrak{sp}(\mathfrak{t}(\kappa))$.

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Chapter 1

Introduction

1.1 Overview

The study of cardinal characteristics of the continuum dates back to at least the beginning of the 20th century and is a cornerstone in the development of set theory. A cardinal characteristic of the continuum is a cardinal lying between ω_1 and 2^ω that captures some combinatorial or topological property of the reals, where the term *the reals* refers to a handful of spaces besides the real line $\mathbb{R}$. Here, we mean either the set of infinite subsets of natural numbers – the set $[\omega]^\omega$ – or the set of functions from the naturals to themselves – the Baire space ${}^\omega\omega$. For notational convenience, we will speak of cardinal characteristics of ${}^\omega\omega$. Usually, such a cardinal characteristic is defined as the minimal cardinality of an object in ${}^\omega\omega$ that does not share a certain specified property of countable objects in ${}^\omega\omega$.

One can of course go beyond the continuum and consider not just the space ${}^\omega\omega$, but the generalized Baire space ${}^\kappa\kappa$, where κ is any regular infinite cardinal. Again, the term *κ-real* refers both to an element of the set ${}^\kappa\kappa$ or to an element of $[\kappa]^\kappa$. As compared to the study of cardinal characteristics of the continuum, research on cardinal characteristics of ${}^\kappa\kappa$ has emerged relatively recently. Its inception, in the 1990's, can be seen in the work on the generalized splitting number $\mathfrak{s}(\kappa)$ by Motoyoshi [31], Kamo [27], Suzuki [38] and Zapletal [40] and in the work of Cummings and Shelah [11] on the generalized bounding and dominating numbers $\mathfrak{b}(\kappa)$ and $\mathfrak{d}(\kappa)$. We begin by formally defining these and the other generalized cardinal characteristics relevant in this thesis.

Definition 1.1. Let κ be a regular infinite cardinal. For $f, g \in {}^\kappa\kappa$, we define $f \leq^* g :\iff |\{\eta \in \kappa : g(\eta) < f(\eta)\}| < \kappa$. For $a, b \in [\kappa]^\kappa$, we define $a \subseteq^* b :\iff |a \setminus b| < \kappa$.

(i) A subset $\mathcal{B} \subseteq {}^\kappa\kappa$ is an *unbounded family* iff there does not exist $f \in {}^\kappa\kappa$ such that for all $g \in \mathcal{B} : g \leq^* f$. Let $\mathfrak{b}(\kappa) := \min\{|\mathcal{B}| : \mathcal{B} \subseteq {}^\kappa\kappa$ is an unbounded family$\}$ be the *κ-bounding number.*

(ii) A subset $\mathcal{D} \subseteq {}^\kappa\kappa$ is a *dominating family* iff for all $f \in {}^\kappa\kappa$ there exists $g \in \mathcal{D}$ such that $f \leq^* g$. Let $\mathfrak{d}(\kappa) := \min\{|\mathcal{D}| : \mathcal{D} \subseteq {}^\kappa\kappa$ is a dominating family$\}$ be the *κ-dominating number.*

(iii) A subset $\mathcal{S} \subseteq [\kappa]^\kappa$ is a *splitting family* iff for all $a \in [\kappa]^\kappa$ there exists $b \in \mathcal{S}$ such that $|a \cap b| = |a \setminus b| = \kappa$. Let $\mathfrak{s}(\kappa) := \min\{|\mathcal{S}| : \mathcal{S} \subseteq [\kappa]^\kappa$ is a splitting family$\}$.

If $\kappa = \omega$, we will usually omit the parentheses in the names of cardinal characteristics, i.e., we will write $\mathfrak{b}$ instead of $\mathfrak{b}(\omega)$.

Note that the property of being an unbounded, dominating or splitting family is closed under taking supersets. This is not the case for the main combinatorial objects studied in this thesis, which are *towers*. For these, as well as for *MAD-families*, which will appear in multiple cameos, it thus makes sense to not just consider the minimal cardinality of which these objects exists, but *all* such cardinalities, which we call their *spectrum.*

Definition 1.2. Let $\kappa \leq \lambda$ be regular cardinals. We call a sequence $\langle a_\xi : \xi \in \lambda\rangle$, where $a_\xi \in [\kappa]^\kappa$, a *$\kappa$-tower of height λ* iff

(i) For all $\xi < \xi' < \lambda : a_\xi \supseteq^* a_{\xi'}$.

(ii) There does not exist an $a \in [\kappa]^\kappa$ with $\forall \xi < \lambda : a_\xi \supseteq^* a$ (no *pseudo-intersection*).

Let $\mathfrak{sp}(\mathfrak{t}(\kappa)) := \{\lambda :$ there exists a κ-tower of height $\lambda\}$ be the *κ-tower spectrum* and $\mathfrak{t}(\kappa) := \min(\mathfrak{sp}(\mathfrak{t}(\kappa)))$ the *κ-tower number.*

Note that this definition excludes towers of non-regular length and of length $<\kappa$, i.e., $\mathfrak{sp}(\mathfrak{t}(\kappa))$ is a set of regular cardinals above κ. This is of course no real restriction since we can always extract a cofinal subsequence from any ordinal-height tower. Conversely, we can always artificially extend a tower as in the definition to an ordinal-height tower by repeating elements. The requirement that $\lambda \geq \kappa$ is a consequence of the following pathology that only arises in the higher Baire spaces:

Fact 1.1. Let κ be regular and uncountable. Decompose κ as $\kappa := \bigcup_{n\in\omega} X_n$, where each X_n has cardinality κ. Then the family $\{\bigcup_{m\geq n} X_m : n \in \omega\}$ is well-ordered by $\supseteq^*$ and has no pseudo-intersection.

We will also study a more restrictive generalization of towers to the higher Baire spaces, namely *club-towers*. Following standard terminology, we say that $c \in [\kappa]^\kappa$ is a *club* iff c contains all of its limit points in κ in the order topology, i.e., iff for all $\alpha \in \kappa : 0 \neq \sup(\alpha \cap c) \in \kappa \implies \sup(\alpha \cap c) \in c$.

Definition 1.3. A *κ-club-tower* is a tower $\langle c_\xi : \xi \in \lambda \rangle$ such that each c_ξ is a club subset of κ. Let $\mathfrak{sp}(\mathfrak{t}_{\mathrm{cl}}(\kappa)) := \{\lambda : \text{there exists a } \kappa\text{-club-tower of height } \lambda\}$ and $\mathfrak{t}_{\mathrm{cl}}(\kappa) := \min(\mathfrak{sp}(\mathfrak{t}_{\mathrm{cl}}(\kappa)))$.

Lastly, we define the generalized MAD-families mentioned above.

Definition 1.4. A family $\mathcal{A} \subseteq [\kappa]^\kappa$ is *almost disjoint* iff for all $a \neq b \in \mathcal{A} : |a \cap b| < \kappa$. Furthermore, $\mathcal{A}$ is maximal almost disjoint (*κ-MAD*) if $\mathcal{A}$ is not properly contained in a different almost disjoint family. Let $\mathfrak{sp}(\mathfrak{a}(\kappa)) := \{\delta : \text{there exists a } \kappa\text{-MAD family } \mathcal{A}$ with $\kappa \leq |\mathcal{A}| = \delta \leq 2^\kappa\}$ be the *κ-MAD spectrum* and $\mathfrak{a}(\kappa) := \min(\mathfrak{sp}(\mathfrak{a}(\kappa)))$ the *κ-maximal almost disjointness number*.

While Cummings and Shelah [11] showed that the behaviour of the two characteristics $\mathfrak{b}(\kappa)$ and $\mathfrak{d}(\kappa)$ mirrors that of their classical counterparts, the value of the generalized splitting number turned out to be linked to various large cardinal assumptions on κ: Motoyoshi [31] showed that $\mathfrak{s}(\kappa) \geq \kappa$ is equivalent to κ being strongly inaccessible, and by a result due to Suzuki [38], $\mathfrak{s}(\kappa) \geq \kappa^+$ is equivalent to κ being weakly compact.[1] Groundbreaking later work by Raghavan and Shelah [32] established that even under such assumptions, the behaviour of the generalized splitting number differs substantially from that of its classical variant. More precisely, $\mathfrak{s}$ and $\mathfrak{b}$ are known to be independent, i.e., both $\mathfrak{s} < \mathfrak{b}$ and $\mathfrak{b} < \mathfrak{s}$ are consistent with ZFC, where the first inequality is due to Baumgartner and Dordal [3] and the second originally to Shelah [36]. In contrast, for uncountable κ, Raghavan and Shelah showed that $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$ is provable in ZFC. An in-depth exposition of these classical results is the topic of Chapter 2 of this thesis.

Further differences between the combinatorics of ${}^\omega\omega$ and those of ${}^\kappa\kappa$ for uncountable κ emerged: Blass, Hyttinen and Zhang [5] proved that if $\mathfrak{d}(\kappa)$ equals κ^+, then so does the characteristic $\mathfrak{a}(\kappa)$. In contrast, the question whether $\mathfrak{d} = \omega_1$ implies $\mathfrak{a} = \omega_1$ is open since the 1970's. Raghavan and Shelah [33] improved the above result and showed that actually, $\mathfrak{b}(\kappa) = \kappa^+$ implies $\mathfrak{a}(\kappa) = \kappa^+$ for uncountable κ. Fischer and

[1] Moreover, Zapletal [40] showed that the existence of a regular uncountable κ with $\mathfrak{s}(\kappa) \geq \kappa^{++}$ has greater consistency strength than the existence of a measurable cardinal.

Soukup established further ZFC inequalities between generalized cardinal characteristics in [18]. Additional recent work on cardinal characteristics of ${}^{\kappa}\kappa$ can be found in [9], [19], [15] and [7].

Generalizing cardinal characteristics to the higher Baire spaces suggests an obvious line of inquiry: Can these characteristics be controlled *globally*, i.e., in all Baire spaces simultaneously? This question can be seen as building upon Easton's famous theorem [14] from the 1960's, showing that the class function $\kappa \to 2^{\kappa}$ on regular κ is essentially independent of ZFC, apart from the obvious restriction that $\kappa < \kappa' \implies 2^{\kappa} \leq 2^{\kappa'}$ and that $\mathrm{cf}(2^{\kappa}) > \kappa$. The first such global result for cardinal characteristics other than 2^{κ} is Cummings' and Shelah's proof [11] that the class function $\kappa \to (\mathfrak{b}(\kappa), \mathfrak{d}(\kappa), 2^{\kappa})$ can be controlled globally, subject to obvious restrictions that mirror the situation at ω, namely,

$$\kappa < \mathfrak{b}(\kappa) = \mathrm{cf}(\mathfrak{b}(\kappa)) \leq \mathrm{cf}(\mathfrak{d}(\kappa)) \leq \mathfrak{d}(\kappa) \leq 2^{\kappa},$$

and with the same conditions on 2^{κ} as in Easton's Theorem.

Our first new result, the topic of Chapter 3, concerns the global separation of $\mathfrak{s}(\kappa)$ and $\mathfrak{b}(\kappa)$. Since $\mathfrak{s}(\kappa)$ is generally smaller than κ by the above discussion and since, by the same argument as in the countable case, $\kappa^{+} \leq \mathfrak{b}(\kappa)$, such a separation is really only interesting if κ is weakly compact. We generalize a result by Bağ and Fischer [1] and show that $\mathfrak{s}(\kappa) = \kappa^{+} < \mathfrak{b}(\kappa)$ is consistent simultaneously for a large class of so-called *strongly unfoldable* cardinals κ. Strongly unfoldable cardinals, originally introduced by Villaveces [39], generalize weakly compact cardinals. The reason for working under the strong-unfoldability assumption instead of the weak-compactness assumption – the former having greater consistency strength than the latter – is that we need to ensure that the forcing extension separating $\mathfrak{s}(\kappa)$ and $\mathfrak{b}(\kappa)$ does not destroy the weak compactness of κ. This is achieved via an indestructibility result by Johnstone [26], allowing us to make strongly unfoldable cardinals indestructible by certain forcing extensions.

The main new results and the focus of this thesis are found in Chapter 4 and deal with the generalized tower spectrum. The ω-tower spectrum has been well-studied for many decades, for example by Hechler [23], by Baumgartner and Dordal [3] or by Dordal [12]. In particular, Hechler [23] showed that consistently, there exists an ω-tower of height λ for each regular $\omega_1 \leq \lambda \leq 2^{\omega}$. Dordal [12, Corollary 2.6] showed that for any set A of regular cardinals containing all of its regular limit points and the successors of its singular limit points, it is consistent that $\mathfrak{sp}(\mathfrak{t}(\omega)) = A$.

The generalized tower number $\mathfrak{t}(\kappa)$ on the other hand was first investigated by Shelah and Spasejović [37]. They showed that $\mathfrak{t}(\kappa) \leq \mathfrak{b}(\kappa)$ for all regular κ and that $\kappa \leq \lambda < \mathfrak{t}(\kappa)$ implies $2^\lambda = 2^\kappa$. Moreover, they proved that for all τ, β, μ with $\kappa < \tau \leq \beta \leq \mu$, such that τ and β are regular and $\mathrm{cf}(\mu) \geq \tau$, it can be forced that $\mathfrak{t}(\kappa) = \tau$, $\mathfrak{b}(\kappa) = \beta$ and $2^\kappa = \mu$. Garti [20] and Fischer et al. [17] investigated whether $\mathfrak{t}(\kappa)$ equals the generalized pseudo-intersection number $\mathfrak{p}(\kappa)$ for uncountable κ, which in the case $\kappa = \omega$ is a famous result due to Malliaris and Shelah [30]. In its full generality, this problem remains unsolved.[2] Further research on the κ-tower number was conducted by Ben-Neria and Garti [4], who proved that for a supercompact κ and for $\kappa < \tau = \mathrm{cf}(\tau) \leq \sigma = \mathrm{cf}(\sigma) \leq \mu$ with $\mathrm{cf}(\mu) > \kappa$, it is consistent that $\mathfrak{t}(\kappa) = \tau$, $\mathfrak{s}(\kappa) = \sigma$ and $2^\kappa = \mu$. Finally, Schilhan [35] observed that for all regular uncountable κ, the characteristic $\mathfrak{t}_{\mathrm{cl}}(\kappa)$ equals $\mathfrak{b}(\kappa)$, which implies that $\mathfrak{b}(\kappa) \in \mathfrak{sp}(\mathfrak{t}(\kappa))$, an important cornerstone for the line of research central to this thesis.

It is folklore that there are no ω-towers of height greater than ω_1 in the Cohen model, which follows from a straightforward isomorphism-of-names argument. We generalize this result and show that in the Easton model, where the class function $\kappa \to 2^\kappa$ can be controlled, the κ-tower spectrum equals $\{\kappa^+\}$ for all regular κ simultaneously. We further show that globally, these small generalized tower spectra are consistent with arbitrarily large generalized MAD spectra, based on previous work by Bağ, Fischer and Friedman [2].

On the other hand, we prove that arbitrarily large tower spectra in all the Baire spaces simultaneously are consistent. In fact, we show that it is consistent that there exists a κ-club-tower of height λ for all regular $\kappa < \lambda \leq 2^\kappa$, where we again have global control over the value of 2^κ.

Finally, we prove that for a fixed regular κ, any upper bound on the κ-tower spectrum is consistent. More concretely, for every regular $\beta > \kappa$ and μ with $\mathrm{cf}(\mu) \geq \beta$, it is consistent that $\mathfrak{b}(\kappa) = \beta$, $2^\kappa = \mathfrak{d}(\kappa) = \mu$, and there are no κ-towers of height greater than β. By Schilhan's result [35] mentioned above, this upper bound is tight for uncountable κ. In case $\kappa = \omega$, the bound is usually also tight, except in the edge case $\beta = \mu$, which will follow from a result in Chapter 2.

These results on the generalized tower spectrum are set to appear in the Journal of Symbolic Logic.

[2] Garti showed that if $\kappa^{<\kappa} = \kappa$ and either $\mathfrak{p}(\kappa) = \kappa^+$ or $\mathrm{cf}(2^\kappa) \in \{\kappa^+, \kappa^{++}\}$, then $\mathfrak{p}(\kappa) = \mathfrak{t}(\kappa)$. Fischer et al. showed that if $\kappa^{<\kappa} = \kappa$, then either $\mathfrak{p}(\kappa) = \mathfrak{t}(\kappa)$, or there exists a $\lambda < \mathfrak{p}(\kappa)$ and a *club-supported* $(\mathfrak{p}(\kappa), \lambda)$*-gap of slaloms*.

1.2 Preliminaries

Combinatorics: As explained previously, a club in κ is a closed unbounded subset of κ. A subset of κ is *stationary* iff it has nonempty intersection with every club in κ. It is well-known that the intersection of fewer than κ many clubs in κ is a club (cf. [25, Theorem 8.3]) and that for any $f \in {}^{\kappa}\kappa$, the set $\{\alpha \in \kappa : f[\alpha] \subseteq \alpha\}$ is a club. More generally, a function $h : \omega \to \omega$ is called *normal* iff h is strictly increasing and for every limit ordinal $\gamma : h(\gamma) = \sup\{h(\alpha) : \alpha \in \gamma\}$. Normal functions have a club set of fixed-points:

Fact 1.2 (Fixed-point Lemma for normal functions, cf. [25, Ex. 8.1])**.** Let $h \in {}^{\kappa}\kappa$ be normal. Then the set $\{\alpha \in \kappa : h(\alpha) = \alpha\}$ is a club in κ.

The following combinatorial lemma is used frequently in this thesis

Fact 1.3 (Δ-system Lemma, cf. [29, Theorem 1.6])**.** Let δ be an infinite cardinal and let $\lambda > \delta$ be regular and such that $\forall \alpha < \lambda : \alpha^{<\delta} < \lambda$. Then, for any $\mathcal{A}$ of cardinality $\geq\lambda$ satisfying $\forall X \in \mathcal{A} : |x| < \delta$, there exists some $\mathcal{A}' \subseteq \mathcal{A}$ of cardinality λ and a *root* R such that $\forall X \neq Y \in \mathcal{A}' : X \cap Y = R$.

Forcing: We follow the convention that stronger forcing conditions are smaller, i.e., $p \leq_{\mathbb{P}} q$ reads "p is stronger than q" or "p extends q". We say that a forcing notion $\mathbb{P}$ satisfies the κ-chain condition (κ-c.c.) iff antichains in $\mathbb{P}$ have cardinality $<\kappa$. The countable chain condition (c.c.c.) is the ω_1-c.c.. Likewise, $\mathbb{P}$ is κ-closed iff descending sequences in $\mathbb{P}$ of length $<\kappa$ have a lower bound. We denote $\mathbb{P}$-names by $\dot{x}$ and canonical $\mathbb{P}$-names for sets in the ground model by $\check{x}$. If there is no danger of confusion, we write just x for the canonical $\mathbb{P}$-name for x in the ground model.

For forcing notions $\mathbb{P}$ and $\mathbb{Q}$, an embedding $i : \mathbb{P} \to \mathbb{Q}$ is a *complete embedding* iff for every maximal antichain A in $\mathbb{P}$, $i[A]$ is a maximal antichain in $\mathbb{Q}$. Equivalently, i preserves incompatibility and for every $q \in \mathbb{Q}$, there exists $p \in \mathbb{P}$ – a *reduction* of q to $\mathbb{P}$ – such that $\forall p' \leq_{\mathbb{P}} p : i(p')$ and q are compatible in $\mathbb{Q}$. If $\mathbb{P} \subseteq \mathbb{Q}$ and the inclusion map is a complete embedding, we say that $\mathbb{P}$ is a *complete suborder* of $\mathbb{Q}$.

For a complete suborder $\mathbb{P} \subseteq \mathbb{Q}$ and a $\mathbb{Q}$ generic filter H over the ground model $\mathbf{V}$, the filter $H \cap \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{V}$. If, on the other hand, G is $\mathbb{P}$-generic over $\mathbf{V}$, we define in $\mathbf{V}[G]$ the *quotient forcing* $\mathbb{Q}/G := \{q \in \mathbb{Q} : \forall p \in G : p$ and q are compatible$\}$. Note that $\mathbb{Q}$ is then forcing equivalent to $\mathbb{P} * \mathbb{Q}/\dot{G}$, where $\mathbb{Q}/\dot{G}$ is a $\mathbb{P}$-name for $\mathbb{Q}/G$. If $\dot{x}$ is a $\mathbb{Q}$-name in the ground model $\mathbf{V}$, there is a $\mathbb{Q}/G$ name $\dot{x}/G \in \mathbf{V}[G]$ – the *quotient name of* $\dot{x}$ – such that for any H that is $\mathbb{Q}/G$-generic over $\mathbf{V}[G]$: $\mathbf{V}[G][H] \models (\dot{x}/G)[H] = \dot{x}[G * H]$.

If $\langle\langle \mathbb{P}_\xi : \xi \leq \alpha\rangle, \langle \dot{\mathbb{Q}}_\xi : \xi < \alpha\rangle\rangle$ is an α-stage iterated forcing construction and G is a $\mathbb{P}_\alpha$-generic filter, we denote by $G|_\beta$ the $\mathbb{P}_\beta$-generic filter $\{p|_\beta : p \in G\}$. We often use the following well-known fact.

Fact 1.4 ([29, Lemma 5.14])**.** Denote by $\mathbf{V}$ the ground model. Assume that α a limit ordinal and that $\langle\langle \mathbb{P}_\xi : \xi \leq \alpha\rangle, \langle \dot{\mathbb{Q}}_\xi : \xi < \alpha\rangle\rangle$ an α-stage iterated forcing with supports in some ideal $\mathcal{I}$, such that each element of $\mathcal{I}$ is bounded in α. If G is $\mathbb{P}_\alpha$-generic over $\mathbf{V}$ and $S \in \mathbf{V}$ is such that $\mathbf{V}[G] \models |S| < \mathrm{cf}(\alpha)$, then, for any $X \in \mathbf{V}[G]$ with $X \subseteq S$, there exists $\beta \in \alpha$ such that $X \in \mathbf{V}[G|_\beta]$.

We denote the support of some $\mathbb{P}_\alpha$-condition p by $\mathrm{supp}(p)$. We use finite support iterations, and, more generally, $<\kappa$-support iterations throughout this thesis. For the global results, we use Easton supports: $\mathbb{P}_\alpha$ is an Easton support iteration or product iff for every regular cardinal $\gamma \leq \alpha$ and every $p \in \mathbb{P}_\alpha : |\mathrm{supp}(p) \cap \gamma| < \gamma$.[3]

Elementary Submodels: Recall that a set model M is an *elementary submodel* of $N \supseteq M$, written $M \prec N$, iff for every formula $\varphi(x_1, ..., x_n)$ and any $a_1, ..., a_n \in M$, we have $M \models \varphi(a_1, ..., a_n) \iff N \models \varphi(a_1, ..., a_n)$. An embedding $j : (M, \in) \to (N, \in)$ is an *elementary embedding* iff $j[M] \prec N$. Note that for transitive M, N, an elementary embedding $j : M \to N$ must map ordinals to ordinals, and if $j \neq \mathrm{id}_M$, there exists a least ordinal δ – the *critical point of* j – such that $j(\delta) > \delta$.

For an infinite cardinal θ, denote by $\mathcal{H}_\theta$ the set $\{x : |\mathrm{trcl}(x)| < \theta\}$, where $\mathrm{trcl}(x)$ is the transitive closure of x. We will frequently use the Löwenheim-Skolem Theorem on $H = \mathcal{H}_\theta$ in this thesis, in the following form.

Fact 1.5 (Löwenheim-Skolem Theorem)**.** Let H be an infinite set model and $A \subseteq H$ an infinite subset. There exists $A \subseteq M \subseteq H$ such that $|M| = |A|$ and $M \prec H$.

Large Cardinals: As explained in the overview, we will encounter a few large cardinals in this thesis, mainly in connection with the generalized splitting number. More concretely, the large cardinal properties we encounter are strong inaccessibility,

[3]Note that this is not the traditional definition of the Easton support. Usually, the condition is that $|\mathrm{supp}(p) \cap \gamma| < \gamma$ for regular *limit* cardinals γ. However, in this thesis, all the iterations/products we consider force nontrivially only at stages that are cardinals, which makes the two definitions equivalent.

weak compactness and strong unfoldability, the last of which we will introduce in Chapter 3, where we need it.

Definition 1.5. A cardinal κ is *strongly inaccessible* iff κ is regular uncountable and for every $\lambda < \kappa : 2^\lambda < \kappa$.

Definition 1.6. A cardinal κ is *weakly compact* iff for every coloring $\pi : [\kappa]^2 \to 2$, there exists some $A \in [\kappa]^\kappa$ such that $\pi|_{[A]^2}$ is constant.

An equivalent definition of weak compactness is given using the *tree property*, which may be viewed as a higher analogue of König's Lemma.

Definition 1.7. A *tree* is a partially ordered set $\langle T, <_T \rangle$ such that for each $x \in T$, the set $\{y \in T : y <_T x\}$ is well-ordered by $<_T$. The *α'th level* of T is the set $\{x \in T : \{y \in T : y <_T x\}$ has order type $\alpha\}$. The *height* of T is the least α for which the α'th level of T is empty. A *branch* of T is a maximal linearly ordered subset of T. Finally, a cardinal κ has the *tree property* iff every tree of height κ whose levels have cardinality $<\kappa$ has a branch of cardinality κ.

Fact 1.6 (cf. [25, Lemma 9.9, Lemma 9.26])**.** κ is weakly compact if and only if it is strongly inaccessible and has the tree property.

Chapter 2

Generalized Splitting and Bounding: Classical Results

2.1 The Characteristics $\mathfrak{s}$ and $\mathfrak{b}$

2.1.1 The Consistency of $\mathfrak{s} < \mathfrak{b}$

The main goal of this first subsection is to construct a model in which $\mathfrak{s} = \omega_1 < \mathfrak{b} = \lambda$, where λ is any regular uncountable cardinal. This result is due to Baumgartner and Dordal [3]. Importantly, it also establishes a fact about ω-towers that will become relevant in later chapters. More concretely, Baumgartner and Dordal use a λ-stage finite support iteration of Hechler forcing, thus increasing $\mathfrak{b}$ while preserving a certain splitting family of cardinality ω_1. This construction will be an edge case of the non-linear iteration of generalized Hechler forcing we use in Chapter 4 in order to bound the κ-tower spectrum from above (Theorem 4.4). We begin by defining the classical, single-stage Hechler forcing.

Definition 2.1. Denote by ${}^{<\omega}\omega{\uparrow}$ the set $\{s \in {}^{<\omega}\omega : s \text{ is strictly increasing}\}$ and define ${}^{\omega}\omega{\uparrow}$ analogously. Hechler forcing $\mathbb{D}$ consists of conditions $\langle s, f\rangle$, where $s \in {}^{<\omega}\omega{\uparrow}$, $f \in {}^{\omega}\omega{\uparrow}$ and

$$\langle s, f\rangle \leq \langle s', f'\rangle :\iff s \supseteq s' \text{ and } f \geq f' \text{ and } \forall n \in \operatorname{dom}(s) \setminus \operatorname{dom}(s') : s(n) > f'(n).$$

It is clear that for every $\mathbb{D}$-generic filter G, the real $g := \bigcup\{s : \exists f\, \langle s, f\rangle \in G\}$ eventually dominates every ground model real. Furthermore, $\mathbb{D}$ satisfies the c.c.c. since conditions with the same first coordinate are compatible. The central ingredient we use in this subsection is the fact that iterations of $\mathbb{D}$ preserve certain splitting families. More precisely, we show that *eventually narrow* sequences are preserved, which implies the preservation of both *eventually splitting* sequences as well as of towers.

Definition 2.2. A sequence $\langle a_\xi : \xi \in \lambda \rangle$ with $a_\xi \in [\omega]^\omega$ is

(i) *eventually splitting* iff

$$\forall a \in [\omega]^\omega \; \exists \xi \in \lambda \; \forall \eta \geq \xi : |a \cap a_\eta| = |a \setminus a_\eta| = \omega,$$

(ii) *eventually narrow* iff

$$\forall a \in [\omega]^\omega \; \exists \xi \in \lambda \; \forall \eta \geq \xi : a \not\subseteq^* a_\eta.$$

Note that $\langle a_\xi : \xi \in \lambda \rangle$ is eventually splitting iff $\langle b_\xi : \xi \in \lambda \rangle$ is eventually narrow, where $b_{2\xi} = a_\xi$ and $b_{2\xi+1} = \omega \setminus a_\xi$. If $\langle a_\xi : \xi \in \lambda \rangle$ is well-ordered by $\supseteq^*$ and λ is regular, it is a tower iff it is eventually narrow. Therefore, it suffices to show that iterations of $\mathbb{D}$ preserve eventually narrow sequences in order to establish the preservation of eventually splitting sequences and towers. Core to the argument that this preservation holds for iterations of $\mathbb{D}$ is the fact that open dense subsets of $\mathbb{D}$ induce a certain rank function on ${}^{<\omega}\omega{\uparrow}$.

Definition 2.3. Let D be an open dense subset of $\mathbb{D}$. Define by induction the following sequence $\langle D_\alpha : \alpha \in \omega_1 \rangle$ in ${}^{<\omega}\omega{\uparrow}$:

$D_0 := \{t \in {}^{<\omega}\omega{\uparrow} : \exists f \, \langle t, f \rangle \in D\}$

$D_{\alpha+1} := \{t \in {}^{<\omega}\omega{\uparrow} :$

(i) Either $t \in D_\alpha$, or

(ii) $\exists l > \mathrm{dom}(t) \; \exists \{t_k : k \in \omega\} \subseteq D_\alpha : \forall k \in \omega \; [t \subseteq t_k \wedge \mathrm{dom}(t_k) = l \wedge t_k(\mathrm{dom}(t)) > k]\}$

$D_\alpha := \bigcup_{\beta \in \alpha} D_\beta$ if α is a limit ordinal.

Note that due to condition (i), the sequence $\langle D_\alpha : \alpha \in \omega_1 \rangle$ is increasing with respect to inclusion. Therefore, since ${}^{<\omega}\omega{\uparrow}$ is countable, there must be a $\gamma_0 \in \omega_1$ such that the sequence stabilizes at γ_0, i.e., $D_{\gamma_0} = D_{\gamma_0+1}$.

Claim 2.1. $D_{\gamma_0} = {}^{<\omega}\omega{\uparrow}$

Proof. Assume by contradiction that there exists an $s \in {}^{<\omega}\omega{\uparrow} \setminus D_{\gamma_0}$. We say that $t \supseteq s$ is a *minimal extension* of s if $t \in D_{\gamma_0}$ and for every $\mathrm{dom}(s) \leq k < \mathrm{dom}(t) : t|_k \notin D_{\gamma_0}$.

Subclaim. For every $n \in \omega$, there are at most finitely many minimal extensions of s of length n, i.e., with domain n.

Proof. Assume that there exists some $n \in \omega$ such that there is an infinite set T of minimal extensions of s of length n. Note that there is some $\mathrm{dom}(s) \leq i < n$ such that $\{t(i) : t \in T\}$ is infinite. Let i_0 be the least such i. Note that $\{t|_{i_0} : t \in T\}$ is finite. By the pigeonhole principle, we find some infinite subset $T' \subseteq T$ and some $u \in {}^{<\omega}\omega{\uparrow}$ such that for all $t \in T' : t|_{i_0} = u$ and such that $\{t(i_0) : t \in T'\}$ is still infinite. We may thus choose $\{t_k : k \in \omega\} \subseteq T'$ such that for all $k \in \omega : t_k(i_0) > k$. This however implies that $u \in D_{\gamma_0+1} = D_{\gamma_0}$ by condition (ii) in the definition of D_{γ_0+1}, which contradicts the fact that the sequences in T' are minimal extensions. $\vdash_{\text{Subclaim}}$

Let T_n be the finite set of minimal extensions of s of length n. Construct a strictly increasing function $f \in {}^{\omega}\omega$ such that for all $\mathrm{dom}(s) \leq n < \omega : f(n) > \max\{t(n) : t \in T_{n+1}\}$. Since D is dense in $\mathbb{D}$, there is some $\langle t, g\rangle \leq \langle s, f\rangle$ with $\langle t, g\rangle \in D$. It follows that $t \supseteq s$ and that $t \in D_0 \subseteq D_\gamma$. Thus, there is some $\mathrm{dom}(s) < i \leq \mathrm{dom}(t)$ such that $t|_i$ is a minimal extension of s. However, this implies that $f(i-1) > t|_i(i-1) = t(i-1)$ by definition of f, which contradicts $\langle t, g\rangle \leq \langle s, f\rangle$. $\vdash_{\text{Claim}}$

Hence, for every open dense $D \subseteq \mathbb{D}$, there is a well-defined rank function on ${}^{<\omega}\omega{\uparrow}$, namely, $\mathrm{rank}_D(s) := \min\{\alpha \in \omega_1 : s \in D_\alpha\}$.

Proposition 2.1 ([3, Theorem 3.1])**.** If $\langle a_\xi : \xi \in \lambda\rangle$ with $\mathrm{cf}(\lambda) > \omega$ is eventually narrow in $\mathbf{V}$, then it remains eventually narrow in any $\mathbb{D}$-generic extension of $\mathbf{V}$.

Proof. Assume by contradiction that there exists a $\mathbb{D}$-name $\dot{x}$ for an infinite subset of ω and some $\mathbb{D}$-condition $\bar{p} = \langle \bar{s}, \bar{f}\rangle$ such that $\bar{p} \Vdash$ "$\forall \xi \in \lambda\ \exists \eta > \xi : \dot{x} \subseteq^* a_\eta$". Let $\dot{h}$ be such that $\Vdash$ "$\dot{h} \in {}^{\omega}\omega$ is the strictly increasing enumeration of $\dot{x}$".

Choose θ large enough so that $\mathbb{D} \in \mathcal{H}_\theta$, and let M be a countable elementary submodel of $\mathcal{H}_\theta$ with $\{\mathbb{D}, \bar{p}, \dot{x}, \dot{h}\} \subseteq M$. Since $\langle a_\xi : \xi \in \lambda\rangle$ is eventually narrow in $\mathbf{V}$, there exists, for every $y \in [\omega]^\omega \cap M$, some $\xi \in \lambda$ such that $\forall \eta \geq \xi : y \not\subseteq^* a_\eta$. Since $[\omega]^\omega \cap M$ is countable and $\mathrm{cf}(\lambda) > \omega$, there is some $\xi_0 \in \lambda$ such that $\forall \eta \geq \xi_0\ \forall y \in [\omega]^\omega \cap M : y \not\subseteq^* a_\eta$. Now, by assumption, $\bar{p}$ forces the existence of some $\eta > \xi_0$ such that $\dot{x} \subseteq^* a_\eta$. Thus, there is some strengthening $\langle s_0, f_0\rangle \leq \bar{p}$, some $\eta_0 \geq \xi_0$ and some $n_0 \in \omega$ such that $\langle s_0, f_0\rangle \Vdash$ "$\forall j \geq n_0 : j \in \dot{x} \implies j \in a_{\eta_0}$".

For every $t \in {}^{<\omega}\omega{\uparrow}$ and every $i \in \omega$, define

$$Z_t(i) := \{j \in \omega : \forall g \in {}^{\omega}\omega{\uparrow}\ \exists q \leq \langle t, g\rangle : q \Vdash \dot{h}(i) = j\}.$$

Note that ${}^{<\omega}\omega{\uparrow} \subseteq M$ and $Z_t(i)$ is definable from parameters in M, which implies that all the $Z_t(i)$ are elements of M.

Claim 2.2. Let t be such that $\langle t, f_0\rangle \leq \langle s_0, f_0\rangle$ and let $i \in \omega$. Then $Z_t(i) \neq \emptyset$.

Proof. Consider the open dense set $D := \{q \in \mathbb{D} : \exists j \in \omega : q \Vdash \text{“}\dot{h}(i) = j\text{”}\}$. We prove the claim by induction on $\operatorname{rank}_D(t)$. If $\operatorname{rank}_D(t) = 0$, then there exists $g \in {}^{\omega}\omega{\uparrow}$ such that $\langle t, g\rangle \Vdash \dot{h}(i) = j$ for some $j \in \omega$. Thus, for any $g' \in {}^{\omega}\omega{\uparrow}$, we can let g'' be the pointwise maximum of g and g' and then $\langle t, g''\rangle$ extends $\langle t, g'\rangle$ and forces "$\dot{h}(i) = j$", which settles this case.

Since $D_\alpha = \bigcup_{\beta\in\alpha} D_\beta$ if α is a limit ordinal, the rank of t must be a successor ordinal. Therefore assume that $t \in D_{\alpha+1} \setminus D_\alpha$. By definition of $D_{\alpha+1}$, there exists some set $\{t_k : k \in \omega\} \subseteq D_\alpha$ of proper extensions of t of length some fixed l, such that for all $k \in \omega :\ t_k(\operatorname{dom}(t)) > k$. By elementarity, such a set also exists in M, therefore we can assume $\{t_k : k \in \omega\} \in M$. Note that for $k \geq k_0 := f_0(l-1)$, $\langle t_k, f_0\rangle$ is a stronger condition than $\langle t, f_0\rangle$ and in particular, $\langle t_k, f_0\rangle \leq \langle s_0, f_0\rangle$, which implies by the induction hypothesis that for these k, $Z_{t_k}(i) \neq \emptyset$.

Subclaim. There exists some $j \in \omega$ such that $j \in Z_{t_k}(i)$ for infinitely many k.

Proof. Assume the contrary. Fix for each $k \geq k_0$ some $j_k \in Z_{t_k}(i)$, for example the minimal one. By assumption, $J := \{j_k : k \geq k_0\}$ is infinite, and since J is defined from parameters in M, we have $J \in M$. Therefore, $J \not\subseteq^* a_{\eta_0}$, which allows us to fix some $k_1 \geq k_0$ such that $j_{k_1} \geq n_0$ and $j_{k_1} \notin a_{\eta_0}$. By definition of $Z_{t_{k_1}}(i)$, there exists $q \leq \langle t_{k_1}, f_0\rangle$ with $q \Vdash \text{“}\dot{h}(i) = j_{k_1}\text{”}$ and therefore $q \Vdash \text{“}j_{k_1} \in \dot{x}\text{”}$. However, $q \leq \langle t_{k_1}, f_0\rangle \leq \langle s_0, f_0\rangle$ and therefore $q \Vdash \text{“}\forall j \geq n_0 : j \in \dot{x} \implies j \in a_{\eta_0}\text{”}$, which implies $j_{k_1} \in a_{\eta_0}$, a contradiction. $\vdash_{\text{Subclaim}}$

By the subclaim, we can fix $j_0 \in \omega$ and $X \in [\omega]^\omega$ such that for all $k \in X : j_0 \in Z_{t_k}(i)$. Let $g \in {}^{\omega}\omega{\uparrow}$. Fix $k \in X$ with $k \geq g(l-1)$, i.e., $\langle t_k, g\rangle \leq \langle t, g\rangle$. By assumption, there exists $q \leq \langle t_k, g\rangle$ with $q \Vdash \text{“}\dot{h}(i) = j_0\text{”}$, and thus $j_0 \in Z_t(i)$. $\vdash_{\text{Claim}}$

By the above claim, $Z_{s_0}(i) \neq \emptyset$ for all $i \in \omega$. Therefore, fix for each $i \in \omega$ some $j_i \in Z_{s_0}(i)$ (for example the minimal one), and let $J' = \{j_i : i \in \omega\}$. Note that $j_i \geq i$ since j_i is forced to be the i'th element of $\dot{x}$. Thus, J' is infinite. Furthermore, since the $Z_{s_0}(i)$ are in M and J' is definable from them, we have $J' \in M$, and therefore $J' \not\subseteq^* a_{\eta_0}$. Fix $j_i \geq n_0$ with $j_i \notin a_{\eta_0}$. There exists $q \leq \langle s_0, f_0\rangle$ forcing "$\dot{h}(i) = j_i$", thus "$j_i \in \dot{x}$", and finally, since $\langle s_0, f_0\rangle \Vdash \text{“}\forall j \geq n_0 : j \in \dot{x} \implies j \in a_{\eta_0}\text{”}$, we obtain the contradiction $q \Vdash \text{“}j_i \in a_{\eta_0}\text{”}$. $\square$

Theorem 2.1 ([3], Theorem 3.3)**.** Let $\mathbb{D}_\alpha$ be the α-stage finite support iteration of $\mathbb{D}$ and $\langle a_\xi : \xi \in \lambda\rangle$ an eventually narrow sequence in the ground model $\mathbf{V}$ with $\mathrm{cf}(\lambda) > \omega$. Then $\langle a_\xi : \xi \in \lambda\rangle$ remains eventually narrow in every $\mathbb{D}_\alpha$-generic extension of $\mathbf{V}$.

Proof. The proof is by induction on α. If α is a successor ordinal, Proposition 2.1 applies. If α is a limit ordinal with $\mathrm{cf}(\alpha) > \omega$, then, for every $\mathbb{D}_\alpha$-generic extension $\mathbf{V}[G]$ and every $x \in [\omega]^\omega \cap \mathbf{V}[G]$, x already appears in $\mathbf{V}[G|_\beta]$ for some $\beta \in \alpha$ by Fact 1.4, and the induction hypothesis applies. Therefore, assume $\mathrm{cf}(\alpha) = \omega$, and let $\langle \beta_n : n \in \omega\rangle$ be a cofinal sequence in α.

Assume by contradiction that there exists a $\mathbb{D}_\alpha$-generic extension $\mathbf{V}[G]$ in which there is some $x \in [\omega]^\omega$ such that $\forall \xi \in \lambda\, \exists \eta > \xi : x \subseteq^* a_\eta$. Let $\dot{x}$ be a $\mathbb{D}_\alpha$-name for x in $\mathbf{V}$. In $\mathbf{V}[G]$, define the set $X := \{\xi \in \lambda : \exists p_\xi \in G\, \exists m_\xi \in \omega : p_\xi \Vdash_{\mathbb{D}_\alpha} \text{“}\dot{x} \setminus m_\xi \subseteq a_\xi\text{”}\}$, which by assumption is cofinal in λ. For each $n \in \omega$, define in $\mathbf{V}[G|_{\beta_n}]$ the set $X_n := \{\xi \in \lambda : \exists p_\xi \in G|_{\beta_n}\, \exists m_\xi \in \omega : p_\xi \Vdash_{\mathbb{D}_\alpha} \text{“}\dot{x} \setminus m_\xi \subseteq a_\xi\text{”}\}$. Since $\mathbb{D}_\alpha$ is a finite support iteration, $X = \bigcup_{n\in\omega} X_n$, which implies by the pigeonhole principle that there exists some fixed $n \in \omega$ such that X_n is cofinal in λ. In $\mathbf{V}[G|_{\beta_n}]$, we again apply the pigeonhole principle to find some fixed $m \in \omega$ and some cofinal $X' \subseteq X_n$ such that for all $\xi \in X' : n_\xi = m$.

It follows in $\mathbf{V}[G|_\beta]$ that $\Vdash_{\mathbb{D}_{\beta_n,\alpha}} \text{“}\forall \xi \in X' : \dot{x}/(G|_{\beta_n}) \setminus m \subseteq a_\xi\text{”}$, where $\mathbb{D}_{\beta_n,\alpha}$ is the quotient $\mathbb{D}_\alpha/(G|_{\beta_n})$ and $\dot{x}/(G|_{\beta_n})$ is the quotient name of $\dot{x}$ in $\mathbf{V}[G|_{\beta_n}]$. Consequently, $\Vdash_{\mathbb{D}_{\beta_n,\alpha}} \text{“}\dot{x}/(G|_{\beta_n}) \setminus m \subseteq \bigcap_{\xi\in X'} a_\xi\text{”}$, which implies in particular that $y := \bigcap_{\xi\in X'} a_\xi$ is infinite. However, since X' is cofinal, we have $\forall \xi \in \lambda\, \exists \eta > \xi : y \subseteq a_\eta$, contradicting the fact that $\langle a_\xi : \xi \in \lambda\rangle$ is eventually narrow in $\mathbf{V}[G|_{\beta_n}]$. □

Corollary 2.1. Forcing with $\mathbb{D}_\alpha$ preserves eventually splitting sequences and towers.

We are now ready to prove the central theorem of this subsection.

Theorem 2.2. Let $\mathbf{V} \models \mathsf{CH}$, let $\lambda > \omega_1$ be regular and let G be $\mathbb{D}_\lambda$-generic over $\mathbf{V}$. Then,

$$\mathbf{V}[G] \models \mathfrak{s} = \omega_1 < \mathfrak{b} = \lambda.$$

Proof. The second equality is easy: Note that every $\mathcal{B} \subseteq {}^\omega\omega$ in $\mathbf{V}[G]$ of cardinality $\delta < \lambda$ already appears in $\mathbf{V}[G|_\alpha]$ for some $\alpha < \lambda$, by letting $h : \delta \to {}^\omega\omega$ be some enumeration of $\mathcal{B}$ and applying Fact 1.4 to the set $\{\langle \xi, n, m\rangle \in \delta \times \omega \times \omega : h(\xi)(n) = m\}$. Hence, $\mathcal{B}$ is eventually dominated by the $\mathbb{D}$-generic real appearing at stage α.[1]

[1]Note that $\mathcal{B} \subseteq \mathbf{V}[G|_\alpha]$ instead of $\mathcal{B} \in \mathbf{V}[G|_\alpha]$ would suffice, and the former can be obtained by applying Fact 1.4 directly to each $f \in \mathcal{B}$ and using the regularity of λ.

This implies $\mathbf{V}[G] \models \mathfrak{b} \geq \lambda$. The reverse inequality follows because $\mathbf{V}[G] \models 2^\omega \leq \lambda$, which can be checked by counting nice names for reals.

For the equality $\mathfrak{s} = \omega_1$, we could use the fact that there is an eventually splitting sequence of length ω_1 in $\mathbf{V}[G|_{\omega_1}]$ since at every earlier stage of the iteration with countable cofinality, a Cohen real is added, and Cohen reals are splitting reals. However, using the fact that the *reaping number* $\mathfrak{r}$ is uncountable and by the CH in $\mathbf{V}$, there is an easier argument.

Lemma 2.1. In $\mathbf{V}$, there is an eventually splitting sequence of length ω_1.

Proof. Let $\langle x_\xi : \xi \in \omega_1 \rangle$ be an enumeration of $[\omega]^\omega$ in $\mathbf{V}$. For every $\zeta \in \omega_1$, choose a_ζ such that a_ζ splits every x_ξ for $\xi \in \zeta$. This is possible because ζ is countable and $\mathfrak{r} \geq \omega_1$, where the reaping number $\mathfrak{r}$ is the minimal cardinality of a subset of $[\omega]^\omega$ not split by any single real (cf. [21, Theorem 9.5]). Now, $\langle a_\zeta : \zeta \in \omega_1 \rangle$ is eventually splitting. □

By Corollary 2.1, this eventually splitting sequence is still eventually splitting in $\mathbf{V}[G]$, which shows that $\mathbf{V}[G] \models \mathfrak{s} = \omega_1$. □

Finally, we prove an additional fact about the λ-stage finite support iteration of Hechler forcing that will be relevant in later chapters.

Theorem 2.3 ([3, Theorem 4.1])**.** Let $\mathbf{V} \models \mathsf{GCH}$, let $\lambda > \omega_1$ be regular and let G be $\mathbb{D}_\lambda$-generic over $\mathbf{V}$. Then, there are no ω-towers of height λ in $\mathbf{V}[G]$.

Proof. Assume towards a contradiction that $\langle a_\xi : \xi \in \lambda \rangle$ is an ω-tower in $\mathbf{V}[G]$. For every $\alpha \in \lambda$ we have $\mathbf{V}[G|_\alpha] \models 2^\omega < \lambda$, which can be seen by counting nice names for subsets of ω. Since $\langle a_\xi : \xi \in \lambda \rangle$ is a tower in $\mathbf{V}[G]$, there exists for each $a \in [\omega]^\omega$ some $\xi(a) < \lambda$ such that $a \not\subseteq^* a_\eta$ for all $\eta \geq \xi(a)$. Therefore, by regularity of λ, we can define in $\mathbf{V}[G]$ for every $\alpha < \lambda : f(\alpha) := \sup\{\xi(a) : a \in [\omega]^\omega \cap \mathbf{V}[G|_\alpha]\} < \lambda$. By the c.c.c. of $\mathbb{D}_\lambda$, there exists $g \in \mathbf{V}$ such that $g(\alpha) > f(\alpha)$ for all $\alpha \in \lambda$.[2] Define the club $C_g := \{\alpha \in \lambda : g[\alpha] \subseteq \alpha\}$.

Claim 2.3. The set $C := \{\alpha \in \lambda : \langle a_\xi : \xi \in \alpha \rangle \in \mathbf{V}[G|_\alpha]\}$ contains a club.

Proof. Let $\dot{T}$ be a $\mathbb{D}_\lambda$-name for $\langle a_\xi : \xi \in \lambda \rangle$ in $\mathbf{V}$. For each $\xi \in \lambda$ and $n \in \omega$, fix a maximal antichain in $\mathbb{D}_\lambda$ deciding "$n \in \dot{T}(\xi)$" and let $A_{\xi,n}$ be its subchain consisting of conditions that force "$n \in \dot{T}(\xi)$". For each $\alpha \leq \lambda$, let $\check{S}_\alpha := \{\langle \mathrm{op}(\xi, \check{a}_\xi), \mathbf{0} \rangle : \xi \in \alpha\}$, where $\check{a}_\xi := \bigcup_{n\in\omega}\{\check{n}\} \times A_{\xi,n}$. Clearly, $\check{S}_\alpha$ is a $\mathbb{D}_\lambda$-name for $\langle a_\xi : \xi \in \alpha \rangle$ in $\mathbf{V}$. By

[2] Let $g(\alpha) := \sup\{\beta \in \omega : \exists p \in \mathbb{D}_\lambda : p \Vdash f(\alpha) = \beta\}$

the c.c.c. of $\mathbb{D}_\lambda$ and since λ is regular, each $\check{S}_\alpha$ is actually already a $\mathbb{D}_\gamma$-name for some $\gamma < \lambda$. Therefore, we can define the function $f : \lambda \to \lambda$, $f(\alpha) := \min\{\gamma \in \lambda : \check{S}_\alpha \text{ is a } \mathbb{D}_\gamma\text{-name}\}$. Since this is a normal function, the set of its fixed-points is a club by Fact 1.2, and if $f(\alpha) = \alpha$, then $\langle a_\xi : \xi \in \alpha \rangle \in \mathbf{V}[G|_\alpha]$. $\vdash_{\text{Claim}}$

It follows that $C_g \cap C$ contains a club. Since $\{\alpha \in \lambda : \text{cf}(\alpha) = \omega_1\}$ is stationary,[3] we can fix $\alpha \in \lambda$ such that $\text{cf}(\alpha) = \omega_1$, $\forall \beta \in \alpha : g(\beta) < \alpha$ and $\langle a_\xi : \xi \in \alpha \rangle \in \mathbf{V}[G|_\alpha]$. It follows that every $a \in [\omega]^\omega \cap \mathbf{V}[G|_\alpha]$ already appears at an earlier stage $\beta < \alpha$ and since $g(\beta) < \alpha$, a is not a pseudo-intersection of $\langle a_\xi : \xi \in \alpha \rangle$. By extracting a cofinal subsequence of α, $\langle a_\xi : \xi \in \alpha \rangle$ is therefore a tower in $\mathbf{V}[G|_\alpha]$, and thus remains a tower in $\mathbf{V}[G]$ by Corollary 2.1, a contradiction. □

2.1.2 The Consistency of $\mathfrak{b} < \mathfrak{s}$

Having established the consistency of $\mathfrak{s} < \mathfrak{b}$, we now prove that the reverse inequality $\mathfrak{b} < \mathfrak{s}$ is consistent as well. This is a result originally due to Shelah [36], who introduced *creature forcing* to obtain a model in which $\mathfrak{b} = \omega_1 < \mathfrak{s} = \omega_2$. We will present a more general result due to Brendle and Fischer [8], showing that for any regular uncountable $\kappa < \lambda$, there is a c.c.c. forcing extension in which $\mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda$. This is done using a *matrix iteration*, a forcing construction originally introduced by Blass and Shelah [6]. Before defining it, we define the forcing notions it will consist of and establish some of their relevant properties.

Definition 2.4. Let $\mathcal{U}$ be an ultrafilter on ω. Mathias forcing $\mathbb{M}_\mathcal{U}$ consists of conditions $\langle a, A \rangle \in [\omega]^{<\omega} \times \mathcal{U}$, where $\max a < \min A$ and

$$\langle a, A \rangle \leq \langle a', A' \rangle :\iff a \supseteq a' \text{ and } A \subseteq A' \text{ and } a \setminus a' \subseteq A'.$$

Since conditions with the same first coordinate are compatible, $\mathbb{M}_\mathcal{U}$ satisfies the c.c.c.. Furthermore, it is clear that forcing with $\mathbb{M}_\mathcal{U}$ adds a pseudo-intersection of $\mathcal{U}$, and thus a real not split by the ground-model reals.

In order to add a MAD-family, we need one half of the forcing notion introduced by Hechler in [23]. A generalization of this forcing to the higher Baire space will appear in Chapter 4.

Definition 2.5. For any ordinal γ, *MAD-Hechler forcing* $\mathbb{A}_\gamma$ consists of conditions $p : F_p \times n_p \to 2$, where $F_p \in [\gamma]^{<\omega}$ and $n_p \in \omega$. Define

$$q \leq p :\iff q \supseteq p \text{ and for all } i \in n_q \setminus n_p : |q^{-1}(1) \cap (F_p \times \{i\})| \leq 1.$$

[3] Take the limit of the first ω_1 many elements of a club.

Lemma 2.2. $\mathbb{A}_\gamma$ satisfies the c.c.c.

Proof. For any uncountable $\mathcal{A} \subseteq \mathbb{A}_\gamma$, there exists $n \in \omega$ such that $n_p = n$ for all $p \in \mathcal{A}$, by the pigeonhole principle. By the Δ-system Lemma, we can extract an uncountable $\mathcal{A}' \subseteq \mathcal{A}$ and a root $R \in [\gamma]^{<\omega}$ such that $F_p \cap F_q = R$ for all $p \neq q \in \mathcal{A}'$. Clearly, the conditions in $\mathcal{A}'$ are pairwise compatible. □

For any $\mathbb{A}_\gamma$-generic G, the family $\{A_\alpha : \alpha \in \gamma\}$ given by $A_\alpha := \{i \in \omega : \exists p \in G : p(\alpha, i) = 1\}$ is almost disjoint by construction. Furthermore, it is maximal almost disjoint if γ is regular uncountable, which can be checked using the c.c.c. of $\mathbb{A}_\gamma$.[4]

Note that $\mathbb{A}_\gamma$ can be decomposed into a two-step iteration as follows:

Definition 2.6. Let $\gamma < \delta$, let G be $\mathbb{A}_\gamma$-generic and let $\{A_\alpha : \alpha \in \gamma\}$ be the generic almost disjoint family induced by G. Let $\mathbb{A}_{[\gamma,\delta)}$ consist of conditions $\langle p, H\rangle$, where

(i) $H \in [\gamma]^{<\omega}$,

(ii) $p : F_p \times n_p \to 2$, with $F_p \in [\delta \setminus \gamma]^{<\omega}$ and $n_p \in \omega$.

The order is given by

$$\langle q, K\rangle \leq \langle p, H\rangle :\Longleftrightarrow \begin{cases} q \leq_{\mathbb{P}_\delta} p \\ K \supseteq H \\ \forall \alpha \in H\ \forall \beta \in F_p\ \forall i \in n_q \setminus n_p : \text{ if } i \in A_\alpha \text{ then } q(\beta, i) = 0. \end{cases}$$

This is clearly isomorphic to the quotient $\mathbb{A}_\delta/G$, and thus $\mathbb{A}_\delta$ decomposes as $\mathbb{A}_\gamma * \dot{\mathbb{A}}_{[\gamma,\delta)}$.

The final forcing notion we need in order to define the matrix iteration is the single stage Hechler forcing $\mathbb{D}$, as defined in Definition 2.1 in the previous subsection. The matrix iteration will be a system of the form

$$\langle\langle \mathbb{P}_{\alpha,\zeta} : \alpha \leq \kappa, \zeta \leq \lambda\rangle, \langle \dot{\mathbb{Q}}_{\alpha,\zeta} : \alpha \leq \kappa, \zeta < \lambda\rangle\rangle,$$

where for each $\alpha \leq \kappa$, the subsystem $\langle\langle \mathbb{P}_{\alpha,\zeta},\ \zeta \leq \lambda\rangle, \langle \dot{\mathbb{Q}}_{\alpha,\zeta} : \zeta < \lambda\rangle\rangle$ is a linear finite support iteration. Moreover, this will be done in such a way that for each $\zeta \leq \lambda$ and all $\beta < \alpha \leq \kappa$, the forcing notion $\mathbb{P}_{\beta,\zeta}$ is a complete suborder of $\mathbb{P}_{\alpha,\zeta}$.

More concretely, we first fix some surjection $f : \{\eta \in \lambda : \eta \equiv 1 \bmod 2\} \to \kappa$ such that $f^{-1}(\alpha)$ is cofinal in λ for every $\alpha \in \kappa$. For a $\mathbb{P}_{\alpha,\zeta}$-generic G, we will denote by $\mathbf{V}_{\alpha,\zeta}$ the forcing extension $\mathbf{V}[G]$. The matrix iteration will be inductively defined as follows:

[4]The skeptical reader is encouraged to consult the proof of Proposition 4.1, where the relevant argument is presented in the context of a different forcing notion.

(i) $\zeta = 0$: For every $\alpha \leq \kappa$, $\mathbb{P}_{\alpha,0} := \mathbb{A}_\alpha$. Thus, forcing with $\mathbb{P}_{\alpha,0}$ will add an almost disjoint family $\{A_\beta : \beta \in \alpha\}$ that is MAD for regular uncountable α. Note that since $\mathbb{A}_\alpha$ decomposes as $\mathbb{A}_\beta * \dot{\mathbb{A}}_{[\beta,\alpha)}$, $\mathbb{P}_{\beta,0}$ will indeed by a complete suborder of $\mathbb{P}_{\alpha,0}$.

(ii) $\zeta = \eta+1$, $\zeta \equiv 1 \bmod 2$: $\mathbb{P}_{\alpha,\zeta} := \mathbb{P}_{\alpha,\eta} * \dot{\mathbb{Q}}_{\alpha,\eta}$, where $\dot{\mathbb{Q}}_{\alpha,\eta}$ is a $\mathbb{P}_{\alpha,\eta}$-name for $\mathbb{M}_{\dot{\mathcal{U}}_{\alpha,\eta}}$, for $\dot{\mathcal{U}}_{\alpha,\eta}$ a $\mathbb{P}_{\alpha,\eta}$-name for a yet to be defined ultrafilter. This ultrafilter will be chosen carefully in order to realize two objectives: Making $\mathbb{P}_{\beta,\zeta}$ a complete suborder of $\mathbb{P}_{\alpha,\zeta}$ for all $\beta < \alpha \leq \kappa$, and establishing the preservation of the MAD-family $\{A_\beta : \beta \in \kappa\}$ added by $\mathbb{P}_{\kappa,0}$. The construction of such an ultrafilter will occupy the bulk of the proof.

(iii) $\zeta = \eta + 1$, $\zeta \equiv 0 \bmod 2$: Again $\mathbb{P}_{\alpha,\zeta} := \mathbb{P}_{\alpha,\eta} * \dot{\mathbb{Q}}_{\alpha,\eta}$. Here, $\dot{\mathbb{Q}}_{\alpha,\eta}$ is a $\mathbb{P}_{\alpha,\eta}$-name for the trivial forcing notion if $\alpha \leq f(\eta)$. If $\alpha > f(\eta)$, $\dot{\mathbb{Q}}_{\alpha,\eta}$ is a $\mathbb{P}_{\alpha,\eta}$-name for $\mathbb{D}^{\mathbf{V}_{f(\eta),\eta}}$.

(iv) ζ is a limit ordinal: For every $\alpha \leq \kappa$, $\mathbb{P}_{\alpha,\zeta}$ is the finite support iteration $\langle\langle \mathbb{P}_{\alpha,\eta},\ \eta \leq \zeta\rangle, \langle \dot{\mathbb{Q}}_{\alpha,\eta} : \eta < \zeta\rangle\rangle$.

Preserving a MAD Family: In order to establish the preservation of MAD-families mentioned in (ii), we define the following property and prove a series of lemmas for it.

Definition 2.7. Let $\mathbf{M} \subseteq \mathbf{N}$ be models of ZFC. Furthermore, let $\mathcal{B} = \{B_\alpha : \alpha \in \gamma\} \subseteq [\omega]^\omega \cap \mathbf{M}$ and $A \in [\omega]^\omega \cap \mathbf{N}$. We say that property $\left(* \,{}^{\mathbf{M},\mathbf{N}}_{\mathcal{B},A} \right)$ holds iff for every $h : \omega \times [\gamma]^{<\omega} \to \omega$ with $h \in \mathbf{M}$ and for every $m \in \omega$, there exists $n \geq m$ and $F \in [\gamma]^{<\omega}$ such that

$$[n, h(n,F)) \setminus \bigcup_{\alpha \in F} B_\alpha \subseteq A.$$

Lemma 2.3. Let $\mathbf{M}$, $\mathbf{N}$, $\mathcal{B} = \{B_\alpha : \alpha \in \gamma\}$ and $A \in [\omega]^\omega$ be such that $\left(* \,{}^{\mathbf{M},\mathbf{N}}_{\mathcal{B},A} \right)$ holds. Denote by $\mathcal{I}$ the ideal generated by $\mathcal{B}$ and the finite sets and let $B \in [\omega]^\omega \cap \mathbf{M}$ be such that $B \notin \mathcal{I}$. Then $\mathbf{N} \models |A \cap B| = \omega$.

Proof. Since $B \notin \mathcal{I}$, we have that for every $n \in \omega$ and every $F \in [\gamma]^{<\omega}$, $B \not\subseteq \bigcup_{\alpha \in F} B_\alpha \cup n$. Thus, there exists $k_{n,F} \geq n$ with $k_{n,F} \in B \setminus \bigcup_{\alpha \in F} B_\alpha$. Define $h(n,F) := k_{n,F} + 1$. By the property $\left(* \,{}^{\mathbf{M},\mathbf{N}}_{\mathcal{B},A} \right)$, there exists for each $m \in \omega$ some $n \geq m$ and some $F \in [\gamma]^{<\omega}$ such that $[n, h(n,F)) \setminus \bigcup_{\alpha \in F} B_\alpha \subseteq A$. In particular, there exists above each $m \in \omega$ some $k_{n,F} \in A \cap B$. □

Lemma 2.3 guarantees that if $\left(* \, {}^{\mathbf{M},\mathbf{N}}_{\mathcal{B},A}\right)$ holds, $\mathcal{B}$ is almost disjoint and B extends $\mathcal{B}$ to a larger almost disjoint family, then B is not almost disjoint from A. The following lemma establishes that this applies to the almost disjoint family introduced by $\mathbb{A}_\gamma$.

Lemma 2.4. Let G be $\mathbb{A}_{\gamma+1}$-generic over $\mathbf{V}$, $G|_\gamma := G \cap \mathbb{A}_\gamma$ and let $\{A_\alpha : \alpha \in \gamma+1\}$ be the almost disjoint family induced by G. Let $\mathcal{A}_\gamma := \{A_\alpha : \alpha \in \gamma\}$ be its $\mathbb{A}_\gamma$-generic subfamily. Then $\left(* \, {}^{\mathbf{V}[G|_\gamma],\, \mathbf{V}[G]}_{\mathcal{A}_\gamma,\, A_\gamma}\right)$ holds.

Proof. Let $h : \omega \times [\gamma]^{<\omega} \to \omega$ with $h \in \mathbf{V}[G|_\gamma]$, let $m \in \omega$ and let $\langle p, H\rangle$ be any $\mathbb{A}_{[\gamma,\gamma+1)}$-condition. We can assume that $\mathrm{dom}(p) = \{\gamma\} \times n_p$. Let $n > \max\{n_p, m\}$. Define $\mathrm{dom}(q) := \{\gamma\} \times n_q$, where $n_q := h(n, H)$, and define

$$q(\gamma, i) := \begin{cases} p(\gamma, i) & \text{if } i \in n_p \\ 1 & \text{if } i \in n_q \setminus n_p \text{ and } i \notin \bigcup_{\alpha \in H} A_\alpha \\ 0 & \text{else.} \end{cases}$$

It follows that $\langle q, H\rangle$ is stronger than $\langle p, H\rangle$ and $\langle q, H\rangle \Vdash_{\mathbb{A}_{[\gamma,\gamma+1)}}$ "$[n, h(n, H)) \setminus \bigcup_{\alpha\in H} A_\alpha \subseteq \dot{A}_\gamma$". □

Next, we prove that $\left(* \, {}^{\mathbf{M},\mathbf{N}}_{\mathcal{B},A}\right)$ is preserved under forcing over $\mathbf{M}$.

Lemma 2.5. Let $\mathbf{M} \subseteq \mathbf{N}$ be models of ZFC, $\mathbb{P} \in \mathbf{M}$ a forcing notion with $\mathbb{P} \subseteq \mathbf{M}$ and $\mathcal{B} = \{B_\alpha : \alpha \in \gamma\} \in \mathbf{M}$, $A \in \mathbf{N}$ such that $\left(* \, {}^{\mathbf{M},\mathbf{N}}_{\mathcal{B},A}\right)$ holds. Then $\left(* \, {}^{\mathbf{M}[G],\mathbf{N}[G]}_{\mathcal{B},A}\right)$ holds for any G that is $\mathbb{P}$-generic over $\mathbf{N}$ (and thus over $\mathbf{M}$).

Proof. Assume by contradiction that there exists $h : \omega \times [\gamma]^{<\omega} \to \omega$ in $\mathbf{M}[G]$ and $m \in \omega$ such that for all $n \geq m$ and $F \in [\gamma]^{<\omega}$: $\mathbf{N}[G] \models [n, h(n, F)) \setminus \bigcup_{\alpha\in F} B_\alpha \nsubseteq A$. Let $\dot{h}$ be a $\mathbb{P}$-name for h in $\mathbf{M}$. There exists $p \in G$ such that, in $\mathbf{N}$, $p \Vdash$ "$\forall n \geq m, \forall F \in [\gamma]^{<\omega} : [n, \dot{h}(n, F)) \setminus \bigcup_{\alpha\in F} B_\alpha \nsubseteq A$". For all n, F, there is some $p_{n,F} \leq p$ and some $k_{n,F} \in \omega$ such that, in $\mathbf{M}$, $p_{n,F} \Vdash$ "$\dot{h}(n, F) = k_{n,F}$". Therefore, again in $\mathbf{N}$, $p_{n,F} \Vdash$ "$[n, k_{n,F}) \setminus \bigcup_{\alpha\in F} B_\alpha \nsubseteq A$", and thus, the function $h_0 : \omega \times [\gamma]^{<\omega} \to \omega$ given by $h_0(n, F) := k_{n,F}$ contradicts $\left(* \, {}^{\mathbf{M},\mathbf{N}}_{\mathcal{B},A}\right)$. □

Similarly, we have the following.

Lemma 2.6. Let ζ be a limit ordinal and for $l \in \{0, 1\}$, let $\mathbb{P}_{l,\zeta}$ be the finite support iteration $\langle\langle \mathbb{P}_{l,\eta},\ \eta \leq \zeta\rangle, \langle \dot{\mathbb{Q}}_{l,\eta} : \eta < \zeta\rangle\rangle$, where $\mathbb{P}_{0,\eta}$ is a complete suborder of $\mathbb{P}_{1,\eta}$ for every $\eta \in \zeta$. Write $\mathbf{V}_{l,\eta}$ for a $\mathbb{P}_{l,\eta}$-generic extension of the ground model $\mathbf{V}_{l,0}$. Then,

(i) $\mathbb{P}_{0,\zeta}$ is a complete suborder of $\mathbb{P}_{1,\zeta}$.

(ii) Let $\mathcal{B} = \{B_\alpha : \alpha \in \gamma\} \in \mathbf{V}_{0,0}$ and $A \in \mathbf{V}_{1,0}$ be such that $\left(* \ {}^{\mathbf{V}_{0,\eta},\mathbf{V}_{1,\eta}}_{\mathcal{B},A} \right)$ holds for every $\eta \in \zeta$. Then $\left(* \ {}^{\mathbf{V}_{0,\zeta},\mathbf{V}_{1,\zeta}}_{\mathcal{B},A} \right)$ holds.

Proof. (i) Clearly, $\mathbb{P}_{0,\zeta} \subseteq \mathbb{P}_{1,\zeta}$ and incompatible conditions in $\mathbb{P}_{0,\zeta}$ remain incompatible in $\mathbb{P}_{1,\zeta}$. For any $\mathbb{P}_{1,\zeta}$-condition p, there exists $\eta \in \zeta$ such that $p \in \mathbb{P}_{1,\eta}$, because p has finite support. Since $\mathbb{P}_{0,\eta}$ is a complete suborder of $\mathbb{P}_{1,\eta}$, there exists a reduction q of p in $\mathbb{P}_{0,\eta}$ (with respect to $\mathbb{P}_{1,\eta}$). However, it is easy to check that q is also a reduction of p with respect to $\mathbb{P}_{1,\zeta}$.

(ii) Assume by contradiction that there exists a $\mathbb{P}_{0,\zeta}$-name $\dot{h}$, a $\mathbb{P}_{1,\zeta}$-condition p and $m \in \omega$ such that $p \Vdash_{\mathbb{P}_{1,\zeta}}$ "$\forall n \geq m, \forall F \in [\gamma]^{<\omega} : [n, \dot{h}(n, F)) \setminus \bigcup_{\alpha \in F} B_\alpha \nsubseteq A$". Since $\mathbb{P}_{1,\zeta}$ is a finite support iteration, there is $\eta \in \zeta$ such that p is in $\mathbb{P}_{1,\eta}$. Let $G_{1,\eta}$ be $\mathbb{P}_{1,\eta}$-generic with $p \in G$ and $G_{0,\eta} := G_{1,\eta} \cap \mathbb{P}_{0,\eta}$. Now, in $\mathbf{V}_{1,\eta} := \mathbf{V}_{1,0}[G_{1,\eta}]$ we have that $\Vdash_{\mathbb{P}_{1,\zeta}/G_{1,\eta}}$ "$\forall n \geq m, \forall F \in [\gamma]^{<\omega} : [n, \ddot{h}(n, F)) \setminus \bigcup_{\alpha \in F} B_\alpha \nsubseteq A$", where $\ddot{h}$ is the quotient name $\dot{h}/G_{0,\eta}$ of $\dot{h}$ in $\mathbf{V}_{0,\eta}$. As in the proof of Lemma 2.5, let $p_{k,F} \in \mathbb{P}_{0,\zeta}/G_{0,\eta}$ be such that $p_{k,F} \Vdash_{\mathbb{P}_{0,\zeta}/G_{0,\eta}} \ddot{h}(n, F) = k_{n,F}$ for some $k_{n,F} \in \omega$. The function $h_0 \in \mathbf{V}_{0,\eta}$ given by $h_0(n, F) := k_{n,F}$ yields a contradiction to $\left(* \ {}^{\mathbf{V}_{0,\eta},\mathbf{V}_{1,\eta}}_{\mathcal{B},A} \right)$. □

Extending the Ultrafilters: The following lemma is the central ingredient needed to define the matrix iteration.

Lemma 2.7. Let $\mathbf{M} \subseteq \mathbf{N}$ be models of ZFC, let $\mathcal{B} = \{B_\alpha : \alpha \in \gamma\} \subseteq [\omega]^\omega \cap \mathbf{M}$ and let $A \in [\omega]^\omega \cap \mathbf{N}$ be such that $\left(* \ {}^{\mathbf{M},\mathbf{N}}_{\mathcal{B},A} \right)$ holds. Let $\mathcal{U}$ be an ultrafilter on ω in $\mathbf{M}$. Then, in $\mathbf{N}$, there is an ultrafilter $\mathcal{V} \supseteq \mathcal{U}$ such that

(i) If C is a maximal antichain of $\mathbb{M}_\mathcal{U}$ in $\mathbf{M}$, then C is a maximal antichain of $\mathbb{M}_\mathcal{V}$ in $\mathbf{N}$.

(ii) $\left(* \ {}^{\mathbf{M}[G],\mathbf{N}[G]}_{\mathcal{B},A} \right)$ holds for any G that is $\mathbb{M}_\mathcal{V}$-generic over $\mathbf{N}$.

Proof. We work in $\mathbf{N}$. We need to extend $\mathcal{U}$ to an ultrafilter in such a way as to avoid *forbidden* subsets of ω, where a subset is forbidden if adding it to the ultrafilter would violate either (i) or (ii) above.

It is clear which subsets must be forbidden in order to avoid property (i) being violated: For $s \in [\omega]^{<\omega}$ and a maximal antichain $C \subseteq \mathbb{M}_\mathcal{U}$ in $\mathbf{M}$, we say that $X \subseteq \omega$ is forbidden by s and C if $\langle s, X\rangle$ is incompatible with every element of C. Crucially, note that incompatibility of $\langle s, X\rangle$ with some $\mathbb{M}_\mathcal{U}$-condition $\langle s', X'\rangle$ does not depend on the yet to be defined ultrafilter $\mathcal{V}$ since this simply means that there is no $s'' \in [\omega]^{<\omega}$ such that $s'' \setminus s \in X$ and $s'' \setminus s' \in X'$.

If property (ii) is violated, there would be some condition $\langle t, Y\rangle$ in the extended Mathias forcing, some $\mathbb{M}_{\mathcal{U}}$ name $\dot{h}$ in $\mathbf{M}$ and some $m \in \omega$ such that $\langle t, Y\rangle$ forces that for every $n \geq m$ and every $F \in [\gamma]^{<\omega}$: "$[n, \dot{h}(n, F)) \setminus \bigcup_{\alpha\in F} B_\alpha \nsubseteq A$". Let $D^{\dot{h}}_{n,F}$ be maximal antichains in $\mathbb{M}_{\mathcal{U}}$ and $g^{\dot{h}}_{n,F} : D^{\dot{h}}_{n,F} \to \omega$ such that for all $p \in D^{\dot{h}}_{n,F}$: $p \Vdash_{\mathbb{M}_{\mathcal{U}}}$ "$\dot{h}(n, F) = g^{\dot{h}}_{n,F}(p)$". We say that $Y \subseteq \omega$ is forbidden by $\dot{h}$ and t as above if for every $n \in \omega$ and $F \in [\gamma]^{<\omega}$: $\langle t, Y\rangle$ is incompatible with every $p \in D^{\dot{h}}_{n,F}$ for which $[n, g^{\dot{h}}_{n,F}(p)) \setminus \bigcup_{\alpha\in F} B_\alpha \subseteq A$.[5] Note that every subset of a forbidden set is also forbidden.

Denote by $\mathcal{I}$ the ideal generated by all the forbidden subsets of ω. Assume $\mathcal{I}\cap\mathcal{U} = \emptyset$, which we prove below. Now, we obtain the desired ultrafilter by applying the Kuratowski-Zorn-Lemma to the set of filter-extensions of $\mathcal{U}$ that are disjoint from $\mathcal{I}$.

Claim 2.4. $\mathcal{I} \cap \mathcal{U} = \emptyset$.

Proof. Assume by contradiction that there are sets X_i, Y_i, $i \in k$, such that X_i is forbidden by s_i and C_i and Y_i is forbidden by $\dot{h}_i$ and t_i and $Z := \bigcup_{i\in k} X_i \cup Y_i \in \mathcal{U}$. We may assume that the X_i and Y_i are all pairwise disjoint since subsets of forbidden sets are forbidden. Fix the following terminology: Some $r \in [\omega]^{<\omega}$ is *permitted* by a Mathias-condition $\langle s, X\rangle$ iff $s \subseteq r$ and $r \setminus s \in X$. Furthermore, r is permitted by an antichain C iff there exists a $p \in C$ that permits r. Note that conditions $\langle s, X\rangle$ and $\langle s', X'\rangle$ are compatible iff there is some $r \in [\omega]^{<\omega}$ permitted by both of them.

Subclaim. In $\mathbf{M}$, there exists $h : \omega \times [\gamma]^{<\omega} \to \omega$ such that for all n, F: $h(n, F) > n$ and for every partition of $Z \cap [n, h(n, F))$ into $2k$ pieces, there exists one piece x such that

(a) For all $i \in k$, there exists $t \subseteq x$ such that C_i permits $s_i \cup t$.

(b) For all $i \in k$, there exists $t \subseteq x$ and some $p \in D^{\dot{h}_i}_{n,F}$ with $g^{\dot{h}_i}_{n,F}(p) < h(n, F)$ that permits $t_i \cup t$.

Proof. We can work in $\mathbf{M}$ since the subclaim only mentions sets in $\mathbf{M}$. Assume by contradiction that there exist n, F such that for all $M > n$, there is a partition of $Z \cap [n, M)$ such that none of its pieces satisfy both (a) and (b). In other words, for every $M > n$, there is a function $f_M : Z \cap [n, M) \to 2k$ such that for all $j \in 2k$: $f^{-1}_M(j)$ does not satisfy both (a) and (b). Note that if either (a) or (b) fails for a piece x, then it fails for every subset of x, thus $f_M|_{Z\cap[n,M')}$ works as a witness for the

[5]Note that this way, every $\langle t, Y\rangle$ that introduces a violation of (ii) through $\dot{h}$ and m is forbidden by $t, \dot{h}'$, where $\dot{h}'$ names the same function as $\dot{h}$ above m and is constantly $\dot{h}(m, F)$ below m.

failure of (a) or (b) for every $M' < M$. It follows that $\{f_M : M > n\}$ is an infinite tree, and thus by König's Lemma, it has an infinite branch $f : Z \setminus n \to 2k$. Now, $\{f^{-1}(j) : j \in 2k\}$ is a partition of the entire $Z \setminus n$ into $2k$ pieces such that none satisfies both (a) and (b).

Since $\mathcal{U}$ is an ultrafilter, one of these pieces X is in $\mathcal{U}$. For every $i \in k$, there is some $p \in C_i$ that is compatible with $\langle s_i, X \setminus (\max s_i)^+ \rangle$ by maximality of C_i. In particular, there is $t \subseteq X$ such that p permits $s_i \cup t$, and analogously, there is some $q \in D^{\dot{h}_i}_{n,F}$ that permits $t_i \cup t'$ for some $t' \subseteq X$. A contradiction. $\vdash_{\text{Subclaim}}$

We continue with the proof of Claim 2.4. Fix h as in the subclaim. For any n,F, consider the partition

$$\{X_i \cap [n, h(n,F)), Y_i \cap [n, h(n,F)) : i \in k\}$$

of $Z \cap [n, F)$. There must be some piece that satisfies (i) and (ii) in the subclaim.

If $X_i \cap [n, h(n,F))$ is that piece, there exists $t \subseteq X_i \cap [n, h(n,F)) \subseteq X_i$ such that $s_i \cup t$ is permitted by some $p \in C_i$, which implies that $\langle s_i, X_i \rangle$ and p are compatible, contradicting the fact that X_i is forbidden by s_i and C_i.

Therefore, some $Y_i \cap [n, h(n,F))$ must be the piece as in the subclaim, meaning that there exists $t \subseteq Y_i \cap [n, h(n,F))$ and $p \in D^{\dot{h}_i}_{n,F}$ with $g^{\dot{h}_i}_{n,F}(p) < h(n,F)$ such that p permits $t_i \cup t$. In particular, p and $\langle t_i, Y_i \rangle$ are compatible. It follows that $[n, g^{\dot{h}_i}_{n,F}(p)) \setminus \bigcup_{\alpha \in F} B_\alpha \nsubseteq A$, by the definition of Y_i being forbidden by $\dot{h}_i$ and t_i. In particular, we have $[n, h(n,F)) \setminus \bigcup_{\alpha \in F} B_\alpha \nsubseteq A$. Since this holds for any n, F, we have a contradiction to property $\left(* \, {}^{\mathbf{M},\mathbf{N}}_{\mathcal{B},A} \right)$. $\vdash_{\text{Claim}}$ $\square$

We need one final well-known fact in order to prove the main theorem of this subsection.

Lemma 2.8 (cf. [8, Lemma 13]). Let $\mathbb{P}$ be a complete suborder of $\mathbb{Q}$, let $\dot{\mathbb{A}}$ be a $\mathbb{P}$-name for a forcing notion and let $\dot{\mathbb{B}}$ be a $\mathbb{Q}$-name for a forcing notion. Furthermore, assume that $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{A}} \subseteq \dot{\mathbb{B}}$ and every maximal antichain of $\dot{\mathbb{A}}$ in $\mathbf{V}[\check{G}|_{\mathbb{P}}]$ is a maximal antichain of $\dot{\mathbb{B}}$ in $\mathbf{V}[\check{G}]$". Then, $\mathbb{P} * \dot{\mathbb{A}}$ is a complete suborder of $\mathbb{Q} * \dot{\mathbb{B}}$.

Theorem 2.4. Let $\kappa < \lambda$ be regular uncountable cardinals in the ground model $\mathbf{V}$. There is a c.c.c. forcing extension of $\mathbf{V}$ in which $\mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda$.

Proof. We recursively define the matrix iteration

$$\langle \langle \mathbb{P}_{\alpha,\zeta} : \alpha \leq \kappa, \zeta \leq \lambda \rangle, \langle \dot{\mathbb{Q}}_{\alpha,\zeta} : \alpha \leq \kappa, \zeta < \lambda \rangle \rangle$$

as described earlier, such that the following two conditions are satisfied, where $\mathcal{A}_\alpha = \{A_\beta,\ \beta \in \alpha\}$ is the $\mathbb{A}_\alpha$-generic almost disjoint family in $\mathbf{V}_{\alpha,0}$.

(i) For every $\zeta \leq \lambda$ and all $\beta < \alpha \leq \kappa$, $\mathbb{P}_{\beta,\zeta}$ is a complete suborder of $\mathbb{P}_{\alpha,\zeta}$.

(ii) For every $\zeta \leq \lambda$ and $\beta < \kappa$, $\left(* \ {}^{\mathbf{V}_{\beta,\zeta},\mathbf{V}_{\beta+1,\zeta}}_{\mathcal{A}_\beta, A_\beta} \right)$ holds.

By recursion on ζ. For $\zeta = 0$ and $\alpha \leq \kappa$, we have already seen that $\mathbb{P}_{\alpha,0} := \mathbb{A}_\alpha$ satisfies (i). On the other hand, property (ii) is satisfied by Lemma 2.4.

Assume that $\zeta = \eta + 1,\ \zeta \equiv 1 \bmod 2$, and $\mathbb{P}_{\alpha,\eta}$ has been defined for all $\alpha \leq \kappa$ in such a way that (i) and (ii) are satisfied. For $\alpha = 0$, let $\dot{\mathcal{U}}_{0,\eta}$ be any $\mathbb{P}_{0,\eta}$-name for an ultrafilter and let $\dot{\mathbb{Q}}_{0,\eta}$ be a $\mathbb{P}_{0,\eta}$-name for $\mathbb{M}_{\dot{\mathcal{U}}_{0,\eta}}$. Define $\mathbb{P}_{0,\zeta} := \mathbb{P}_{0,\eta} * \dot{\mathbb{Q}}_{0,\eta}$. We proceed by induction and first assume that $\alpha = \beta + 1$. By Lemma 2.7, there is a $\mathbb{P}_{\alpha,\eta}$-name $\dot{\mathcal{U}}_{\alpha,\eta}$ for an ultrafilter such that

$$\Vdash_{\mathbb{P}_{\alpha,\eta}} \begin{cases} \text{(a) } "\dot{\mathcal{U}}_{\beta,\eta} \subseteq \dot{\mathcal{U}}_{\alpha,\eta}" \text{ and} \\ \text{(b) } "\text{every maximal antichain of } \mathbb{M}_{\dot{\mathcal{U}}_{\beta,\eta}} \text{ in } \mathbf{V}_{\beta,\eta} \text{ is a maximal antichain of } \mathbb{M}_{\dot{\mathcal{U}}_{\alpha,\eta}} \text{ in } \mathbf{V}_{\alpha,\eta}" \text{ and} \\ \text{(c) } "\left(* \ {}^{\mathbf{V}_{\beta,\zeta},\mathbf{V}_{\alpha,\zeta}}_{\mathcal{A}_\beta, A_\beta} \right) \text{ holds", where } \mathbf{V}_{\alpha,\zeta} \text{ is a } \mathbb{M}_{\dot{\mathcal{U}}_{\alpha,\eta}}\text{-generic extension of } \mathbf{V}_{\alpha,\eta}. \end{cases}$$

Let $\dot{\mathbb{Q}}_{\alpha,\eta}$ be a $\mathbb{P}_{\alpha,\eta}$-name for $\mathbb{M}_{\dot{\mathcal{U}}_{\alpha,\eta}}$ and set $\mathbb{P}_{\alpha,\zeta} := \mathbb{P}_{\alpha,\eta} * \dot{\mathbb{Q}}_{\alpha,\eta}$. By Lemma 2.8, $\mathbb{P}_{\beta,\zeta}$ is a complete suborder of $\mathbb{P}_{\alpha,\zeta}$. Condition (ii) is satisfied due to (c).

If α is a limit ordinal, consider the filter named by $\bigcup_{\beta<\alpha} \dot{\mathcal{U}}_{\beta,\eta}$. If $\mathrm{cf}(\alpha) > \omega$, this is an ultrafilter and (b) is satisfied for every $\beta < \alpha$ since every real in $\mathbf{V}_{\alpha,\eta}$ already appears in some earlier $\mathbf{V}_{\beta,\eta}$. In the case $\mathrm{cf}(\alpha) = \omega$, we need to extend it to an ultrafilter first, such that (i) and (ii) are satisfied, using similar arguments as in Lemma 2.7. For the details, see Blass and Shelah [6, p. 266].

If $\zeta = \eta + 1,\ \zeta \equiv 0 \bmod 2$, we define $\mathbb{P}_{\alpha,\zeta} = \mathbb{P}_{\alpha,\eta} * \dot{\mathbb{Q}}_{\alpha,\eta}$ as described earlier: If $\alpha \leq f(\eta)$, let $\dot{\mathbb{Q}}_{\alpha,\eta}$ be a $\mathbb{P}_{\alpha,\eta}$-name for the trivial forcing notion and if $\alpha > f(\eta)$, let $\dot{\mathbb{Q}}_{\alpha,\eta}$ be a $\mathbb{P}_{\alpha,\eta}$-name for $\mathbb{D}^{\mathbf{V}_{f(\eta),\eta}}$. It is easy to see that this satisfies (i): If $\beta < \alpha \leq f(\eta)$, then $\mathbb{P}_{\beta,\zeta} = \mathbb{P}_{\beta,\eta}$ and $\mathbb{P}_{\alpha,\zeta} = \mathbb{P}_{\alpha,\eta}$, which yields (i) by the induction hypothesis. If $\beta \leq f(\eta) < \alpha$, then $\mathbb{P}_{\beta,\zeta} = \mathbb{P}_{\beta,\eta}$ is a complete suborder of $\mathbb{P}_{\alpha,\eta}$ by the induction hypothesis and thus of $\mathbb{P}_{\alpha,\eta} * \dot{\mathbb{Q}}_{\alpha,\eta} = \mathbb{P}_{\alpha,\zeta}$. If $f(\eta) < \beta < \alpha$, then $\mathbb{P}_{\beta,\zeta}$ is a complete suborder of $\mathbb{P}_{\alpha,\zeta}$ by Lemma 2.8. Similarly, (ii) follows from the induction hypothesis for $\beta < f(\eta)$ and from Lemma 2.5 for $\beta \geq f(\eta)$.

Finally, if ζ is a limit ordinal, we let $\mathbb{P}_{\alpha,\zeta}$ be the finite support iteration $\langle\langle \mathbb{P}_{\alpha,\eta},\ \eta \leq \zeta\rangle, \langle \dot{\mathbb{Q}}_{\alpha,\eta} : \eta < \zeta\rangle\rangle$. Then, (i) and (ii) are satisfies by Lemma 2.6.

Claim 2.5. Let $\zeta \leq \lambda$. Then,

(i) For every $\mathbb{P}_{\kappa,\zeta}$-condition p, there exists $q \leq_{\mathbb{P}_{\kappa,\zeta}} p$ and $\alpha < \kappa$ such that q is already a $\mathbb{P}_{\alpha,\zeta}$-condition.

(ii) If $\dot{f}$ is a $\mathbb{P}_{\kappa,\zeta}$-name for a real, there exists $\alpha < \kappa$ and a nice $\mathbb{P}_{\alpha,\zeta}$-name $\dot{g}$ such that $\Vdash_{\mathbb{P}_{\kappa,\zeta}} \dot{f} = \dot{g}$.

Proof. By induction on ζ. Note that (ii) follows from (i) since $\mathbb{P}_{\kappa,\zeta}$ satisfies the c.c.c. and κ is regular. If $\zeta = 0$, then (i) follows directly from the definition of $\mathbb{A}_\kappa$. If $\zeta = \eta+1$ and $p \in \mathbb{P}_{\kappa,\zeta}$, then p is of the form $p_0 {}^\frown \dot{q}$, where p_0 is a $\mathbb{P}_{\kappa,\eta}$-condition and $\dot{q}$ is a $\mathbb{P}_{\kappa,\eta}$-name for either a Mathias condition or a Hechler condition. By strengthening p_0 to p_1, we can assume that $\dot{q}$ is of the form $\mathrm{op}(\check{s}, \dot{x})$, where $s \in [\omega]^{<\omega}$ or $s \in {}^{<\omega}\omega$ and $\dot{x}$ is a name for an element of either $[\omega]^\omega$ or ${}^\omega\omega$. In either case, by the induction hypothesis, there is some $\alpha < \kappa$ such that $\dot{x}$ is equivalent to a nice $\mathbb{P}_{\alpha,\eta}$-name $\dot{y}$. Also by the induction hypothesis, there exists $p_2 \leq_{\mathbb{P}_{\kappa,\eta}} p_1$ such that p_2 is a $\mathbb{P}_{\alpha',\eta}$-condition for some $\alpha' < \kappa$. It follows that $p_2 {}^\frown \mathrm{op}(\check{s}, \dot{y})$ is in $\mathbb{P}_{\max\{\alpha,\alpha'\},\zeta}$.

Finally, if ζ is a limit ordinal and $p \in \mathbb{P}_{\kappa,\zeta}$, then p already lies in $\mathbb{P}_{\kappa,\eta}$ for some $\eta < \zeta$ since $\mathbb{P}_{\kappa,\zeta}$ is a finite support iteration. By the induction hypothesis, there exists $q \leq_{\mathbb{P}_{\kappa,\eta}} p$ that is in $\mathbb{P}_{\alpha,\eta}$ for some $\alpha < \kappa$. Hence, q is in $\mathbb{P}_{\alpha,\zeta}$. $\dashv_{\text{Claim}}$

Claim 2.6. $\mathbf{V}_{\kappa,\lambda} \models \mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda$.

Proof. In $\mathbf{V}_{\kappa,\lambda}$, we have $\mathfrak{b} \leq \mathfrak{a} \leq \kappa$ since $\mathfrak{b} \leq \mathfrak{a}$ is a well-known ZFC inequality[6] and because the almost disjoint family $\{A_\alpha : \alpha \in \kappa\}$ remains MAD in $\mathbf{V}_{\kappa,\lambda}$: For every $B \in [\omega]^\omega \cap \mathbf{V}_{\kappa,\lambda}$, there exists $\alpha < \kappa$ such that $B \in \mathbf{V}_{\alpha,\lambda}$, by Claim 2.5. Either B has infinite intersection with one of the $A_\beta \in \mathcal{A}_\alpha$, or B is not in the ideal $\mathcal{I}$ generated by $\mathcal{A}_\alpha$ and the finite sets, and therefore $|B \cap A_\alpha| = \omega$ by Lemma 2.3.

For any $\mathcal{B} \subseteq {}^\omega\omega \cap \mathbf{V}_{\kappa,\lambda}$ of cardinality $<\kappa$, there exists again by Claim 2.5 and by regularity of κ some $\alpha < \kappa$ and $\zeta < \lambda$ such that $\mathcal{B} \subseteq {}^\omega\omega \cap \mathbf{V}_{\alpha,\zeta}$. Since $f^{-1}(\alpha)$ is cofinal in λ, there is some $\zeta' \geq \zeta$ such that $f(\zeta') = \alpha$. Thus, $\mathbb{P}_{\alpha+1,\zeta'+1} = \mathbb{P}_{\alpha+1,\zeta'} * \dot{\mathbb{Q}}_{\alpha+1,\zeta'}$, where $\dot{\mathbb{Q}}_{\alpha+1,\zeta'}$ is a $\mathbb{P}_{\alpha+1,\zeta'}$-name for $\mathbb{D}^{\mathbf{V}_{\alpha,\zeta'}}$. It follows that there is a real x in $\mathbf{V}_{\alpha+1,\zeta'+1}$ dominating ${}^\omega\omega \cap \mathbf{V}_{\alpha,\zeta'}$, and thus in particular $\mathcal{B}$. Consequently, $\mathbf{V}_{\kappa,\lambda} \models \mathfrak{b} \geq \kappa$, which by the above yields $\mathbf{V}_{\kappa,\lambda} \models \mathfrak{b} = \mathfrak{a} = \kappa$.

Finally, every $\mathcal{S} \subseteq [\omega]^\omega \cap \mathbf{V}_{\kappa,\lambda}$ of cardinality $<\lambda$ already appears in $\mathbf{V}_{\kappa,\eta}$ for some $\eta < \lambda$ with $\eta \equiv 0 \bmod 2$. The subsequent Mathias forcing then adds a real not split by $[\omega]^\omega \cap \mathbf{V}_{\kappa,\eta}$. On the other hand, since $\mathbb{D}$ adds dominating reals, it adds splitting reals, which yields a splitting family of cardinality λ. Thus $\mathbf{V}_{\kappa,\lambda} \models \mathfrak{s} = \lambda$. $\dashv_{\text{Claim}}$ $\square$

[6] see, for example, [21, Theorem 9.7].

2.2 The Generalized Characteristics $\mathfrak{s}(\kappa)$ and $\mathfrak{b}(\kappa)$

2.2.1 $\mathfrak{s}(\kappa)$ and Large Cardinals

Before proving that the independence of $\mathfrak{s}$ and $\mathfrak{b}$ fails in the higher Baire spaces, we show how the value of $\mathfrak{s}(\kappa)$ is linked to two large cardinal assumptions on κ. In particular, the inequality $\mathfrak{s}(\kappa) \geq \kappa^+$, which we might expect to hold for any cardinal characteristic, is equivalent to κ being weakly compact. We first give a proof that $\mathfrak{s}(\kappa) \geq \kappa$ holds if and only if κ is strongly inaccessible, a result originally due to Motoyoshi [31].

Theorem 2.5 ([31], cf. [40, Lemma 3])**.** $\mathfrak{s}(\kappa) \geq \kappa$ is equivalent to κ being strongly inaccessible.

Proof. First, assume that κ is not strongly inaccessible. Let $\lambda < \kappa$ be the least cardinal with $2^\lambda \geq \kappa$. Fix some injection $\varphi : \kappa \to {}^\lambda 2$ and define for each $s \in {}^{<\lambda}2$ the set $b_s := \{\alpha \in \kappa : s \subseteq \varphi(\alpha)\}$. Note that the family $\mathcal{S} := \{b_s : s \in {}^{<\lambda}2\} \cap [\kappa]^\kappa$ has cardinality $<\kappa$, by the minimality of λ. We show that $\mathcal{S}$ is a splitting family.

Assume by contradiction that there exists $y \in [\kappa]^\kappa$ not split by $\mathcal{S}$. Define $X := \{s \in {}^{<\lambda}2 : |b_s \cap y| = \kappa\}$. By assumption, we have $b_{s_0} \cap b_{s_1} \neq \emptyset$ for all $s_0, s_1 \in X$, and hence, for every $\alpha \in b_{s_0} \cap b_{s_1} : s_0, s_1 \subseteq \varphi(\alpha)$, showing that X is linearly ordered by "$\subseteq$". Consequently, there exists $f \in {}^\lambda 2$ such that all $s \in X$ are initial segments of f, which implies that for every $\alpha \in y \setminus \varphi^{-1}(f)$, there is an initial segment t of $\varphi(\alpha)$ such that $t \notin X$. Thus, $y \setminus \varphi^{-1}(f) \subseteq \bigcup\{b_t \cap a : t \in {}^{<\lambda}2 \setminus X\}$. However, the latter is a union of $<\kappa$ many sets of cardinality $<\kappa$ and $y \setminus \varphi^{-1}(f)$ has cardinality κ.

For the other direction, we use the following lemma, which we will reuse in the proof of Theorem 2.6.

Lemma 2.9. Assume that κ is strongly inaccessible and that $\mathcal{S} \subseteq [\kappa]^\kappa$ is of cardinality $<\kappa$. There exists a decomposition $\mathcal{S} = B_0 \cup B_1$ such that the set $C := \bigcap B_0 \setminus \bigcup B_1$ has cardinality κ

Proof. Let $\mathcal{S} \subseteq [\kappa]^\kappa$ be of cardinality $\lambda < \kappa$. For a sufficiently large θ, let M be an elementary submodel of $\mathcal{H}_\theta$, such that $\kappa \in M$, $\mathcal{P}(\mathcal{S}) \subseteq M$ and $|M| = 2^\lambda < \kappa$. Let $\sup(\kappa \cap M) < \delta < \kappa$ and define $B_0 := \{x \in \mathcal{S} : \delta \in x\}$ and $B_1 := \{x \in \mathcal{S} : \delta \notin x\}$. Note that $B_0, B_1 \in M$ since $\mathcal{P}(\mathcal{S}) \subseteq M$. In particular, it follows that $C \in M$. Since

$\delta \in C$, we have for every $\alpha \in \kappa \cap M$: $\mathcal{H}_\theta \models \exists \beta \in C : \beta > \alpha$, and therefore, since M is an elementary submodel of $\mathcal{H}_\theta$, the set $C \cap M$ is unbounded in $\kappa \cap M$. Thus, again by elementarity, $|C| = \kappa$. $\square$

Now, let $\mathcal{S} \subseteq [\kappa]^\kappa$ be of cardinality $<\kappa$ and B_0, B_1, C as in the lemma. For every $x \in \mathcal{S}$, either $x \in B_1$, in which case $C \subseteq x$, or $x \in B_2$, in which case $C \cap x = \emptyset$. In both cases, x does not split C, and hence $\mathcal{S}$ is not a splitting family. $\square$

Next, we prove the main result of this subsection, the equivalence of $\mathfrak{s}(\kappa) \geq \kappa^+$ and weak compactness of κ. This is originally due to Suzuki [38]. The proof presented here is based both on Raghavan and Shelah [32] and on Zapletal [40].

Theorem 2.6 ([38, Theorem 1], cf. [32, Lemma 3.1], [40, Lemma 4])**.** $\mathfrak{s}(\kappa) \geq \kappa^+$ if and only if κ is weakly compact.

Proof. Assume that $\mathfrak{s}(\kappa) \geq \kappa^+$ and let $\pi : [\kappa]^2 \to 2$ be a coloring. We need to find some $A \in [\kappa]^\kappa$ that is monochromatic with respect to π.

Define for each $\alpha \in \kappa$ and $j \in 2$ the set $K_{\alpha,j} := \{\beta > \alpha : \pi(\{\alpha, \beta\}) = j\}$. By assumption, $\{K_{\alpha,0} : \alpha \in \kappa\}$ is not a splitting family, hence there exists some $x \in [\kappa]^\kappa$ such that for all $\alpha \in \kappa$, either $x \subseteq^* K_{\alpha,0}$ or $x \subseteq^* K_{\alpha,1}$. By the pigeonhole principle, we find $y \subseteq x$ of cardinality κ such that, without loss of generality, for all $\alpha \in y : x \subseteq^* K_{\alpha,0}$. We define $A = \{\gamma_\alpha : \alpha \in \kappa\} \subseteq y$ by induction as follows: Suppose $\langle \gamma_\xi : \xi \in \alpha \rangle \in {}^\alpha y$ is defined. For every $\xi \in \alpha$, let $\delta_\xi \in \kappa$ be minimal such that $x \setminus \delta_\xi \subseteq K_{\gamma_\xi,0}$ and $\delta_{\xi'} < \delta_\xi$ for all $\xi' < \xi$. Let $\delta := \sup\{\delta_\xi : \xi \in \alpha\}$ and set $\gamma_\alpha := \min y \setminus \delta$.

Clearly, $\alpha < \beta$ implies $\gamma_\alpha < \gamma_\beta$, and thus, $A \in [\kappa]^\kappa$. Furthermore, by construction, $\pi(\gamma_\alpha, \gamma_\beta) = 0$ for all $\alpha < \beta < \kappa$, and hence, A is monochromatic.

We prove the other direction via the tree property. Assume κ is weakly compact and let $\mathcal{S} = \{a_\xi : \xi \in \kappa\}$ be of cardinality κ. We show that $\mathcal{S}$ is not a splitting family. Let $T \subseteq {}^{<\kappa}2$ be the following tree:

$$T := \bigcup_{\alpha \in \kappa} \{t \in {}^\alpha 2 : \text{ The set } C_t := \bigcap_{\substack{\xi \in \alpha: \\ t(\xi)=1}} a_\xi \setminus \bigcup_{\substack{\xi \in \alpha: \\ t(\xi)=0}} a_\xi \text{ has cardinality } \kappa\}.$$

Since κ is inaccessible, each level of T has cardinality at most $2^\alpha < \kappa$. Furthermore, by Lemma 2.9 in the proof of Theorem 2.5, for every $\alpha \in \kappa$ there exists $t \in T \cap {}^\alpha 2$, and hence, T has height κ. Now, by the weak compactness of κ, T has a cofinal branch $f : \kappa \to 2$.

By induction, construct a strictly increasing sequence $\langle \beta_\alpha : \alpha \in \kappa \rangle$ with $\beta_\alpha \in C_{f|\alpha}$ and set $C := \{\beta_\alpha : \alpha \in \kappa\}$. For any $\xi \in \kappa$, either $\{\beta_\alpha : \alpha > \xi\} \subseteq a_\xi$ if $f(\xi) = 1$ or $\{\beta_\alpha : \alpha > \xi\} \cap a_\xi = \emptyset$ if $f(\xi) = 0$. In both cases, a_ξ does not split C, showing that $\mathcal{S}$ is not a splitting family. □

2.2.2 ZFC proves $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$ for Regular Uncountable κ

The following observation is due to Schilhan [35]. Apart from allowing us to simplify Raghavan's and Shelah's proof of the main theorem of this subsection, it will be important in Chapter 4.

Lemma 2.10 ([35, Theorem 2.5, Theorem 2.9])**.** Let κ be regular uncountable. Then $\mathfrak{b}(\kappa) = \mathfrak{t}_{\mathrm{cl}}(\kappa)$.

Proof. First, let $\mathcal{B} \subseteq {}^\kappa\kappa$ be an unbounded family of cardinality $\mathfrak{b}(\kappa)$. We may assume without loss of generality that $\mathcal{B}$ is well-ordered by $\leq^*$, i.e., $\mathcal{B} = \{g_\xi : \xi \in \mathfrak{b}(\kappa)\}$ and $\xi < \xi' \implies g_\xi \leq^* g_{\xi'}$. Furthermore, we can assume that the $g_\xi \in \mathcal{B}$ are strictly increasing. Consider the sequence of clubs $\langle c_\xi : \xi \in \mathfrak{b}(\kappa) \rangle$, where $c_\xi := \{\alpha \in \kappa : g_\xi[\alpha] \subseteq \alpha\}$. It is easy to check that this sequence is well-ordered by $\supseteq^*$. Furthermore, if $p \in [\kappa]^\kappa$ were a pseudo-intersection of the c_ξ, the function $f_p \in {}^\kappa\kappa$ given by $f_p(\alpha) := \min(p \setminus (\alpha + 1))$ would dominate $\mathcal{B}$.

On the other hand, for any set $\mathcal{H}$ of clubs without a pseudo-intersection, the family $\{f_c : c \in \mathcal{H}\} \subseteq {}^\kappa\kappa$ is unbounded since if $g \in {}^\kappa\kappa$ were dominating, the club $\{\alpha \in \kappa : g[\alpha] \subseteq \alpha\}$ would pseudo-intersect $\mathcal{H}$. □

In particular, Lemma 2.10 implies that $\mathfrak{b}(\kappa) \in \mathfrak{sp}(\mathfrak{t}(\kappa))$ for regular uncountable κ. While not necessary for the proof of the main theorem of this subsection, the following lemma provides a corresponding result for the case $\kappa = \omega$.

Lemma 2.11 (Folklore)**.** Assume $\mathfrak{b}(\omega) < \mathfrak{d}(\omega)$. Then $\mathfrak{b}(\omega) \in \mathfrak{sp}(\mathfrak{t}(\omega))$.

Proof. Let $\mathcal{B} = \{g_\xi : \xi \in \mathfrak{b}(\omega)\} \subseteq {}^\omega\omega$ be unbounded and such that $\xi < \xi' \implies g_\xi \leq^* g_{\xi'}$. Assume further that every $g_\xi \in \mathcal{B}$ is strictly increasing. Since $\mathfrak{b}(\omega) < \mathfrak{d}(\omega)$, there exists $f \in {}^\omega\omega$ that is not dominated by $\mathcal{B}$. For each $\xi \in \mathfrak{b}(\omega)$, let $a_\xi := \{n \in \omega : f(n) > g_\xi(n)\}$. Clearly, the sequence $\langle a_\xi : \xi \in \mathfrak{b}(\omega) \rangle$ is well-ordered by $\supseteq^*$. If it were pseudo-intersected by $p \in [\omega]^\omega$, the function $f_p \in {}^\omega\omega$ given by $f_p(n) := f(\min(p \setminus (n + 1)))$ would dominate $\mathcal{B}$. □

Note that the assumption $\mathfrak{b}(\omega) < \mathfrak{d}(\omega)$ is necessary: The model constructed in Section 2.1.1 satisfies $\mathfrak{b}(\omega) = \mathfrak{d}(\omega) = 2^\omega = \lambda$, but contains no ω-tower of height λ by Theorem 2.3. Next, we prove the main theorem of this subsection, Raghavan's and Shelah's surprising discovery that the higher characteristics $\mathfrak{s}(\kappa)$ and $\mathfrak{b}(\kappa)$ are not independent.

Theorem 2.7 (cf. [32, Theorem 1.10])**.** For κ regular uncountable, $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$.

Proof. Assume by contradiction that $\mathfrak{b}(\kappa) < \mathfrak{s}(\kappa)$, i.e., by Lemma 2.10, $\kappa < \mathfrak{b}(\kappa) = \mathfrak{t}_{\mathrm{cl}}(\kappa) =: \lambda < \mathfrak{s}(\kappa)$. Let $\langle c_\xi : \xi \in \lambda \rangle$ be a κ-club-tower of height λ. For a large enough θ, let M be an elementary submodel of $\mathcal{H}_\theta$ such that $\lambda \subseteq M$, $\{c_\xi : \xi \in \lambda\} \subseteq M$ and $|M| = \lambda$. By assumption, $[\kappa]^\kappa \cap M$ is not a splitting family, hence there exists some $A_0 \in [\kappa]^\kappa$ such that for all $x \in [\kappa]^\kappa \cap M : A_0 \subseteq^* x$ or $A_0 \subseteq^* \kappa \setminus x$. Define the filter $\mathcal{F} := \{x \in [\kappa]^\kappa \cap M : A_0 \subseteq^* x\}$, which is an ultrafilter over M. Note that $\mathcal{F}$ is κ-complete, i.e., intersections of fewer than κ many elements of $\mathcal{F}$ are in $\mathcal{F}$. We use the following lemma, which is similar to a result by Scott (cf. [28, Ex. 5.12]).

Lemma 2.12. There exists $f_* \in {}^\kappa\kappa \cap M$ such that for every club $C \in [\kappa]^\kappa \cap M$: $f_*^{-1}(C) \in \mathcal{F}$.

We prove this lemma below. Assuming we have it, we find that for each $\xi < \lambda : f_*^{-1}(c_\xi) \in \mathcal{F}$, and thus $A_0 \subseteq^* f_*^{-1}(c_\xi)$. Hence, for all $\xi \in \lambda : f_*(A_0) \subseteq^* c_\xi$, which contradicts the assumption that $\langle c_\xi : \xi \in \lambda \rangle$ is a tower, provided $f_*(A_0)$ is of cardinality κ. But this is easy to see because for every $\beta \in \kappa$, the set $\kappa \setminus \beta$ is a club and an element of M (since $\kappa \subseteq M$), which implies by the Lemma that $f_*^{-1}(\kappa \setminus \beta) \in \mathcal{F}$. Therefore, $A_0 \subseteq^* f_*^{-1}(\kappa \setminus \beta)$, and in particular there exists $\alpha \in A_0$ with $f_*(\alpha) > \beta$.

Proof of Lemma 2.12. Define the following equivalence relation on ${}^\kappa\kappa \cap M$:

$$\forall f, g \in {}^\kappa\kappa \cap M : f \sim_{\mathcal{F}} g :\iff \{\alpha \in \kappa : f(\alpha) = g(\alpha)\} \in \mathcal{F}.$$

Denote the equivalence class of f by $[f]_{\mathcal{F}}$ and set $L := \{[f]_{\mathcal{F}} : f \in {}^\kappa\kappa \cap M\}$. Define the following order on L.

$$[f]_{\mathcal{F}} <_{\mathcal{F}} [g]_{\mathcal{F}} :\iff \{\alpha \in \kappa : f(\alpha) < g(\alpha)\} \in \mathcal{F}.$$

Since $\mathcal{F}$ is a filter, this is well-defined and $<_{\mathcal{F}}$ is transitive. Denote by $i \in {}^\kappa\kappa$ the identity function on κ and for each $\alpha \in \kappa$, let $c_\alpha \in {}^\kappa\kappa$ be the function that is constantly equal to α. Note that the c_α and i are in M, since $\kappa \cup \{\kappa\} \subseteq M$.

For any $\delta < \kappa$ and any partition $\langle X_\xi : \xi \in \delta \rangle \in M$ of κ, there is a unique $\iota \in \delta$ with $X_\iota \in \mathcal{F}$: Since $\delta \subseteq M$, each X_ξ is in M, and hence each $\kappa \setminus X_\xi$ is in M. If none of the X_ξ were in $\mathcal{F}$, each $\kappa \setminus X_\xi$ would be and thus, by κ-completeness of $\mathcal{F}$, it would follow that $\emptyset = \bigcap_{\xi \in \delta} \kappa \setminus X_\xi \in \mathcal{F}$.

Claim 2.7. L is well-ordered by $<_\mathcal{F}$.

Proof. For all $[f]_\mathcal{F}, [g]_\mathcal{F} \in L$, the sets $\{\alpha \in \kappa : f(\alpha) = g(\alpha)\}$, $\{\alpha \in \kappa : f(\alpha) < g(\alpha)\}$ and $\{\alpha \in \kappa : f(\alpha) > g(\alpha)\}$ are in M and partition κ. Hence, exactly one of them is in $\mathcal{F}$, which shows that $<_\mathcal{F}$ is a linear ordering. To check that $<_\mathcal{F}$ is a well-ordering, assume by contradiction that $\langle [f_j]_\mathcal{F} : j \in \omega \rangle$ is strictly descending with respect to $<_\mathcal{F}$. Thus, for each $j \in \omega$, the set $X_j := \{\alpha \in \kappa : f_{j+1}(\alpha) < f_j(\alpha)\}$ is in $\mathcal{F}$. Since $\kappa > \omega$, the set $X := \bigcap_{j \in \omega} X_j$ is in $\mathcal{F}$ by κ-completeness.[7] In particular $X \neq \emptyset$. Hence, for $\beta \in X$, the sequence $\langle f_j(\beta) : j \in \omega \rangle$ is a strictly descending sequence of ordinals, which is a contradiction. $\dashv_{\text{Claim}}$

In particular, note that $[i]_\mathcal{F}$ is an upper bound for the set $\{[c_\alpha]_\mathcal{F} : \alpha \in \kappa\}$, and hence, since $<_\mathcal{F}$ is a well-ordering, $\{[c_\alpha]_\mathcal{F} : \alpha \in \kappa\}$ has a least upper bound $[f_*]_\mathcal{F}$. It remains to prove that f_* is the required function, i.e., that $f_*^{-1}(C) \in \mathcal{F}$ for every club $C \in [\kappa]^\kappa \cap M$.

Assume by contradiction that for some $C \in [\kappa]^\kappa \cap M$, $f_*^{-1}(C) \notin \mathcal{F}$. Since f_*, C and κ are in M, $f_*^{-1}(C)$ and $\kappa \setminus f_*^{-1}(C) = f_*^{-1}(\kappa \setminus C)$ are as well, and thus, $Y := f_*^{-1}(\kappa \setminus C) \in \mathcal{F}$. Define $f : \kappa \to \kappa$, $f(\alpha) := \sup(C \cap f_*(\alpha))$ and note $f \in M$. For every $\alpha \in Y$, $f_*(\alpha) \notin C$, and therefore, since C is closed, $f(\alpha) < f_*(\alpha)$, which shows that $[f]_\mathcal{F} <_\mathcal{F} [f_*]_\mathcal{F}$. Since C is unbounded, there exists for each $\beta \in \kappa$ some $\delta > \beta$ with $\delta \in C$, and because $[c_\delta]_\mathcal{F} <_\mathcal{F} [f_*]_\mathcal{F}$, the set $Z := \{\alpha \in \kappa : f_*(\alpha) > \delta\}$ is in $\mathcal{F}$. For each $\alpha \in Z : c_\beta(\alpha) = \beta < \delta \leq f(\alpha)$, which shows $[c_\beta]_\mathcal{F} <_\mathcal{F} [f]_\mathcal{F}$. Hence, $[f]_\mathcal{F}$ is an upper bound for $\{[c_\beta]_\mathcal{F} : \beta \in \kappa\}$ and $[f]_\mathcal{F} <_\mathcal{F} [f_*]_\mathcal{F}$, a contradiction. □ □

[7] Note that $\kappa > \omega$ is necessary here. For $\kappa = \omega$, a strictly descending sequence as above is given by setting $f_j(n) := n - j$ for $n \geq j$ and $f_j(n) := 0$ otherwise.

Chapter 3

Separating $\mathfrak{s}(\kappa)$ and $\mathfrak{b}(\kappa)$ at Large Cardinals

Having established that $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$ for uncountable κ, the main goal of this chapter is to show that the separation $\mathfrak{s}(\kappa) < \mathfrak{b}(\kappa)$ is consistent at many κ simultaneously. Stated in this form, this is of course trivial since by Theorem 2.6, $\mathfrak{s}(\kappa) \leq \kappa < \mathfrak{b}(\kappa)$ if κ is not weakly compact. Hence, we want a model in which $\mathfrak{s}(\kappa) < \mathfrak{b}(\kappa)$ for many weakly compact κ simultaneously. As explained previously, we assume a slightly stronger large cardinal property of κ – namely strong unfoldability – in order to ensure that κ remains weakly compact after forcing the separation. The definition of strong unfoldability is somewhat technical.

Definition 3.1. Denote by ZFC^- the axioms of ZFC without the power set axiom and let θ be an ordinal. A cardinal κ is θ-strongly unfoldable iff κ is inaccessible and for every transitive set model $M \models \mathsf{ZFC}^-$ with $|M| = \kappa$, $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$, there exists a transitive set model N and an elementary embedding $j : M \to N$ with critical point κ such that $\theta < j(\kappa)$ and $\mathbf{V}_\theta \subseteq N$. A cardinal κ is strongly unfoldable iff it is θ-strongly unfoldable for every ordinal θ.

The following fact is due to Villaveces [39].

Fact 3.1 ([39, Proposition 1.6])**.** Every strongly unfoldable cardinal is weakly compact.

Relevant for our purposes is Johnstones indestructibility result [26] mentioned in the overview. In order to state it, we need to define the notion of a *κ-proper forcing.*

Definition 3.2. Assume that κ is a cardinal with $\kappa^{<\kappa} = \kappa$. A forcing notion $\mathbb{P}$ is *κ-proper* iff for all sufficiently large regular λ, there is an $x \in \mathcal{H}_\lambda$ such that for every elementary submodel X of $\mathcal{H}_\lambda$ with $|X| = \kappa$, ${}^{<\kappa}X \subseteq X$ and $\{\kappa, \mathbb{P}, x\} \subseteq X$, there exists for every $p \in \mathbb{P} \cap X$ an *X-generic* $q \leq p$. Recall that q is *X-generic* iff every $\mathbb{P}$-generic filter G over $\mathbf{V}$ with $q \in G$ intersects every dense set $D \in X$ in X, meaning $G \cap D \cap X \neq \emptyset$.

Fact 3.2 ([26, Fact 13.4])**.** Every κ^+-c.c. forcing notion is κ-proper.

Here is the crucial theorem. We provide a sketch of its proof below.

Theorem 3.1 (Johnstone [26, Main Theorem])**.** For every strongly unfoldable cardinal κ, there exists a forcing notion $\mathbb{P}$, the *lottery preparation of* κ, such that forcing with $\mathbb{P}$ makes the strong unfoldability of κ indestructible by every κ-closed, κ-proper forcing notion. Furthermore, $\mathbb{P}$ satisfies the κ-c.c. and $\mathbb{P} \in \mathcal{H}_{\kappa^+}$.

Thus, precomposing the forcing notion separating $\mathfrak{s}(\kappa)$ and $\mathfrak{b}(\kappa)$ – which will be κ-closed and satisfy the κ^+-c.c. – with the lottery preparation, we can ensure that κ remains strongly unfoldable and hence weakly compact in the extension. In fact, it follows from [26, Fact 25] that such a preparatory forcing is necessary: Should we happen to force directly over the ground model $\mathbf{L}$, the weak compactness of κ would indeed be destroyed in the extension.

The subsequent few pages sketch the proof of Theorem 3.1. In order to construct a model in which $\mathfrak{s}(\kappa) < \mathfrak{b}(\kappa)$ only at a single cardinal κ, which was done by Bağ and Fischer [1] (cf. Section 3.1), Theorem 3.1 can be applied as a black box[1], and the reader interested in that result may therefore skip ahead to Section 3.1. For the global separation (Section 3.2) however, some of the details in the construction of the lottery preparation are relevant.

Proof of Theorem 3.1 (Sketch). Fix the following terminology: For a strongly inaccessible κ, any transitive $M \models \mathsf{ZFC}^-$ with $|M| = \kappa$, $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$ is called a *κ-model*. Furthermore, an elementary embedding $j : M \to N$ with critical point κ such that $\theta < j(\kappa)$ and $\mathbf{V}_\theta \subseteq N$ is called a *θ-strong unfoldability embedding*.

As noted above, $\mathbb{P}$ is the lottery preparation of κ, a tool developed by Hamkins in [22]. For a collection $\mathcal{A}$ of forcing notions, denote by $\bigoplus \mathcal{A}$ the forcing notion $\{\langle \mathbb{Q}, p\rangle : \mathbb{Q} \in \mathcal{A},\ p \in \mathbb{Q}\} \cup \{\mathbb{1}\}$, where $\mathbb{1}$ is the maximal element and $\langle \mathbb{Q}, p\rangle \leq \langle \mathbb{Q}', p'\rangle \iff \mathbb{Q} = \mathbb{Q}'$ and $p \leq_{\mathbb{Q}} p'$. Since a $\bigoplus \mathcal{A}$-generic filter is simply a $\mathbb{Q}$-generic

[1] In fact, so can the definition of strong unfoldability itself.

filter for a 'randomly' selected $\mathbb{Q} \in \mathcal{A}$, the forcing notion $\bigoplus \mathcal{A}$ is called the *lottery sum* of $\mathcal{A}$. With this lottery sum, we define the lottery preparation of κ relative to some fixed (partial) function $f \in {}^\kappa\kappa$: It is the length κ iteration with Easton support which at stage $\gamma \in \kappa$, if $\gamma \in \mathrm{dom}(f)$ and $f[\gamma] \subseteq \gamma$, consists of the lottery sum of all the γ-closed forcing notions $\mathbb{Q} \in \mathcal{H}_{f(\gamma)^+}$ in a $\mathbb{P}_\gamma$-generic extension. All other stages of the lottery preparation are trivial.

The lottery preparation used in the proof is relative to an $f \in {}^\kappa\kappa$ with the so called *Menas property for* κ. This means that for every ordinal θ and every κ-model M with $f \in M$, there exists a θ-strong unfoldability embedding $j : M \to N$ such that $j(f)(\kappa) \geq \beth_\theta^N$. We omit the proof that a function with the Menas property for κ exists.

Fix any $f \in {}^\kappa\kappa$ with the Menas property for κ, let $\mathbb{P}$ be the lottery preparation of κ relative to f and $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name for a κ-closed, κ-proper forcing notion. Furthermore, fix an ordinal θ. We need the following alternative characterisation of θ-strong unfoldability:

Fact 3.3 ([26, Fact 4])**.** A cardinal κ is θ-strongly unfoldable iff for every $A \subseteq \kappa$, there exists a κ-model M and a θ-strong unfoldability embedding $j : M \to N$ such that $A \in M$.

Hence, to show that κ remains θ-strongly unfoldable in any $\mathbb{P} * \dot{\mathbb{Q}}$-generic extension, we need to show that for any subset A of κ in the extension, the set of $\mathbb{P} * \dot{\mathbb{Q}}$-conditions forcing the existence of a κ-model containing A and of a θ-strong unfoldability embedding of this κ-model, are dense in $\mathbb{P} * \dot{\mathbb{Q}}$. Let $\dot{A}$ be a $\mathbb{P} * \dot{\mathbb{Q}}$-name for a subset of κ and r' any $\mathbb{P} * \dot{\mathbb{Q}}$-condition.

First, note that one can prove that $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-proper. Choose λ large enough so that $\mathcal{H}_\lambda$ witnesses this and contains all the relevant sets. Let $X \prec \mathcal{H}_\lambda$ be of size κ such that ${}^{<\kappa}X \subseteq X$ and $\{\kappa, r', \mathbb{P}, f, \dot{\mathbb{Q}}, \dot{A}, \theta\} \subseteq X$. By κ-properness, there is an X-generic $r \leq r'$. Let $G * g$ be $\mathbb{P} * \dot{\mathbb{Q}}$-generic over $\mathbf{V}$ with $r \in (G * g)$. The claim is that r is the required condition, i.e., that in $\mathbf{V}[G * g]$, the set $A := \dot{A}[G]$ can be placed in a κ-model with a corresponding embedding.

Let $\pi : (X, \in) \to (M, \in)$ be the Mostowski-collapse of X, so that M is a κ-model. By induction, $\pi|_{\mathbf{V}_\kappa} = \mathrm{id}$, which implies that π fixes κ, $\mathbb{P}$ and f, but generally collapses $\dot{\mathbb{Q}}$ to $\pi(\dot{\mathbb{Q}}) =: \dot{\mathbb{Q}}_0$ and $\dot{A}$ to $\pi(\dot{A}) =: \dot{A}_0$. Since κ is $(\theta+1)$-strongly unfoldable, there exists a $(\theta+1)$-strong unfoldability embedding $j : M \to N$. By a result in [13], we may assume that $|N| = \beth_{\theta+1}$ and that ${}^{\beth_\theta}N \subseteq N$. Furthermore, by possibly including an additional forcing, Johnstone shows that we may assume $\beth_{\theta+1} = \beth_\theta^+$.

From the fact that $G * g$ is X-generic – since it contains r – it follows with a bit of work that the pointwise image $\pi[G * g]$ is of the form $G * g_0$, where G is M-generic for $\mathbb{P}$ and g_0 is $M[G]$-generic for $\mathbb{Q}_0 := \dot{\mathbb{Q}}_0[G]$. Crucially, it further follows that $A = \dot{A}_0[G * g_0]$. Thus, $A \in M[G * g_0]$, which gives us a candidate κ-model containing A.

The core of the proof now consist in finding a θ-strong unfoldability embedding of $M[G * g_0]$. This is done by lifting the embedding $j : M \to N$ twice – first to an embedding $j : M[G] \to N[j(G)]$ and then the latter to an embedding $j : M[G][g_0] \to N[j(G)][j(g_0)]$. By the *lifting criterion*, this succeeds if and only if there exists, firstly, an N-generic filter $j(G)$ for the forcing notion $j(\mathbb{P})$ such that $j[G] \subseteq j(G)$, and secondly, an $N[j(G)]$-generic filter $j(g_0)$ for $j(\mathbb{Q}_0)$ such that $j[g_0] \subseteq j(g_0)$.

To construct the filter $j(G)$ for $j(\mathbb{P})$, note that in N, $j(\mathbb{P})$ is the lottery preparation of $j(\kappa) > \kappa$ relative to $j(f)$. Since f has the Menas property, $j(f)(\kappa)$ is large enough so that at stage κ of the iteration $j(\mathbb{P})$, the forcing notion $\mathbb{Q}$ appears in the lottery sum. Therefore, by constructing the filter above a condition that selects $\mathbb{Q}$ as the 'winning' forcing notion at stage κ, we can decompose $j(\mathbb{P})$ as $\mathbb{P} * \dot{\mathbb{Q}} * \mathbb{P}_{\text{tail}}$. Since, by elementarity, we already have an N-generic filter for $\mathbb{P} * \dot{\mathbb{Q}}$, namely $G * g$, we only need to construct an $N[G * g]$-generic filter G_{tail} for $\mathbb{P}_{\text{tail}}$ and set $j(G) := G * g * G_{\text{tail}}$. Such a G_{tail} can be constructed by diagonalization, similarly to how generic filters for *countable* transitive models are constructed: One notices that $|N[G * g]| \leq \beth_\theta^+$, that ${}^{\beth_\theta}N[G * g] \subseteq N[G * g]$ and crucially, that $\mathbb{P}_{\text{tail}}$ is $\beth_\theta^+$-closed, which follows from the definition of the lottery preparation and the fact that $j(f)(\kappa) \geq \beth_\theta$ by the Menas property.

It remains to lift the embedding $j : M[G] \to N[j(G)]$ to $j : M[G * g_0] \to N[j(G) * j(g_0)]$, again using the lifting criterion. This is easy because we can show that there exists a $j(\mathbb{Q}_0)$-condition q above $j[g_0]$. Thus, by constructing $j(g_0)$ again by diagonalization and in such a way that $q \in j(g_0)$, we get a filter with $j[g_0] \subseteq j(g_0)$.

Finally, note that $G * g \in N[j(G)]$ by construction of $j(G)$ and that $\mathbf{V}_{\theta+1} \subseteq N$, by definition of a $(\theta + 1)$-strong unfoldability embedding. This implies

$$\mathbf{V}[G * g]_{\theta+1} \subseteq \mathbf{V}_{\theta+1}[G * g] \subseteq N[j(G) * j(g_0)],$$

which shows that the twice lifted j is indeed a θ-strong unfoldability embedding of $M[G * g_0]$. $\square$

3.1 $\mathfrak{s}(\kappa) = \kappa^+ < \mathfrak{b}(\kappa)$ at a Single κ

The construction separating $\mathfrak{s}(\kappa)$ and $\mathfrak{b}(\kappa)$ globally is based on a construction at a single cardinal κ due to Bağ and Fischer [1], which we present in this section. Probably the most intuitive approach to obtain a model in which $\mathfrak{s}(\kappa) = \kappa^+ < \mathfrak{b}(\kappa) = \lambda$ would be a λ-stage iteration of κ-Hechler forcing (the generalization of Hechler forcing to κ), analogous to how $\mathfrak{s} = \omega_1 < \mathfrak{b} = \lambda$ was obtained in Section 2.1.1. While generalized eventually narrow sequences are indeed preserved by the single stage generalized Hechler forcing (cf. [1, Theorem 3.5]), the preservation fails for longer iterations at stages of countable cofinality. This can also be seen in the following way: If eventually narrow sequences were preserved by iterations of κ-Hechler forcing of length $\lambda > \kappa^+$, the argument in the proof of Theorem 2.3 would go through, which would show that there are no κ-towers of height $\lambda = \mathfrak{b}(\kappa)$ in the extension. However, there must exist such a tower by Lemma 2.10.

Hence, a different strategy is needed. By Lemma 2.10, we know that $\mathfrak{b}(\kappa) = \mathfrak{t}_{\mathrm{cl}}(\kappa)$, which suggests iteratively pseudo-intersecting the set of κ-clubs in order to increase $\mathfrak{b}(\kappa)$. This is achieved using the following variant of Mathias forcing.

Definition 3.3. Denote by $\mathcal{C}$ the set of all clubs in κ. The forcing notion $\mathbb{M}(\mathcal{C})$ consists of pairs $\langle a, C\rangle$, where $a \in [\kappa]^{<\kappa}$ and $C \in \mathcal{C}$, ordered by $\langle a, C\rangle \leq (a', C') :\iff a \supseteq a'$, $C \subseteq C'$ and $a \setminus a' \subseteq C'$. Let $\mathbb{M}(\mathcal{C})_\lambda$ be the λ-stage, $<\kappa$-support iteration of $\mathbb{M}(\mathcal{C})$.

It is clear that for every $\mathbb{M}(\mathcal{C})$-generic filter G, the κ-real $\bigcup\{a : \exists C \in \mathcal{C} : \langle a, C\rangle \in G\}$ pseudo-intersects every ground model club. We can assume without loss of generality that $\mathbb{M}(\mathcal{C})_\lambda$-conditions are of the form $\langle \bar{a}, \bar{C}\rangle$, where

(i) $\bar{a}$ is a sequence $\langle a_\xi : \xi \in I\rangle$ with $I \in [\lambda]^{<\kappa}$ and $a_\xi \in [\kappa]^{<\kappa}$,

(ii) $\bar{C}$ is a sequence $\langle \dot{C}_\xi : \xi \in I\rangle$, where $\dot{C}_\xi$ is an $\mathbb{M}(\mathcal{C})_\xi$-name for a club in κ.

Fact 3.4. Let κ be regular uncountable, $\kappa^{<\kappa} = \kappa$ and $\lambda > 0$ any ordinal.

(i) $\mathbb{M}(\mathcal{C})_\lambda$ is κ-closed.

(ii) $\mathbb{M}(\mathcal{C})_\lambda$-conditions $p_0 = \langle \bar{a}_0, \bar{C}_0\rangle$ and $p_1 = \langle \bar{a}_1, \bar{C}_1\rangle$ for which

$$\forall \xi \in \mathrm{dom}(\bar{a}_0) \cap \mathrm{dom}(\bar{a}_1) : \ \bar{a}_0(\xi) = \bar{a}_1(\xi)$$

are compatible.

(iii) $\mathbb{M}(\mathcal{C})_\lambda$ satisfies the κ^+-c.c..

Proof. Fact (i) holds because the single stage forcing $\mathbb{M}(\mathcal{C})$ is κ-closed in the obvious way, by the regularity of κ. Since we force with $<\kappa$-supports, $\mathbb{M}(\mathcal{C})_\lambda$ is also κ-closed. Fact (ii) is also checked easily since for any $\mathbb{M}(\mathcal{C})_\xi$-names $\dot{C}_1$ and $\dot{C}_2$ for clubs, we can build an $\mathbb{M}(\mathcal{C})_\xi$-name $\dot{C}$ for a club such that $\Vdash_{\mathbb{M}(\mathcal{C})_\xi}$ "$\dot{C} = \dot{C}_1 \cap \dot{C}_2$". For Fact (iii), let A be a set of $\mathbb{M}(\mathcal{C})_\lambda$-conditions with $|A| = \kappa^+$. For $p = \langle \bar{a}_p, \bar{C}_p \rangle$, set $I_p := \mathrm{dom}(\bar{a}_p) = \mathrm{dom}(\bar{C}_p)$. Applying the Δ-system Lemma, we obtain $A' \subseteq A$ of cardinality κ^+ and $R \in [\lambda]^{<\kappa}$ such that for all $p, p' \in A'$: $I_p \cap I_{p'} = R$. By the assumption $\kappa^{<\kappa} = \kappa$ and the pigeonhole principle, there exists $\langle r_\xi : \xi \in R \rangle$ with $r_\xi \in [\kappa]^{<\kappa}$ and some $A'' \subseteq A'$ of cardinality κ^+ such that for all $p \in A''$ and $\xi \in R$: $\bar{a}_p(\xi) = r_\xi$. By (ii), the conditions in A'' are pairwise compatible. □

Definition 3.4. By induction on λ, define nice $\mathbb{M}(\mathcal{C})_\lambda$-names for subsets of κ as follows: Assume we have defined nice $\mathbb{M}(\mathcal{C})_\alpha$-names for all $\alpha \in \lambda$. A nice $\mathbb{M}(\mathcal{C})_\lambda$-name $\dot{x}$ is any name of the form $\dot{x} = \bigcup_{\xi \in \kappa} \{\check{\xi}\} \times A_\xi$, where A_ξ is an antichain in $\mathbb{M}(\mathcal{C})_\lambda$ and for each $\langle \bar{a}, \bar{C} \rangle \in A_\xi$ and each $\zeta \in \mathrm{dom}(\bar{C})$, $\bar{C}(\zeta)$ is a nice $\mathbb{M}(\mathcal{C})_\zeta$-name for a club in κ.

Fact 3.5. For any $\mathbb{M}(\mathcal{C})_\lambda$-name $\dot{y}$ for a subset of κ, there exists a nice $\mathbb{M}(\mathcal{C})_\lambda$-name $\dot{x}$ such that $\Vdash \dot{y} = \dot{x}$.

Definition 3.5. Denote by $\mathbb{C}_{\kappa^+}$ the $<\kappa$-support product of κ^+ many κ-Cohen forcings $\mathbb{C} = \mathrm{Fn}_{<\kappa}(\kappa, 2)$.[2] Let $\lambda > \kappa^+$ be regular and denote by $\mathbb{Q}$ the forcing notion $\mathbb{C}_{\kappa^+} * \dot{\mathbb{M}}(\mathcal{C})_\lambda$.

Lemma 3.1. Assume $\kappa^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$.

(i) $\mathbb{Q}$ is κ-closed and satisfies the κ^+-c.c.,

(ii) $\Vdash_{\mathbb{Q}} \mathfrak{b}(\kappa) = \mathfrak{t}_{cl}(\kappa) = 2^\kappa = \lambda$.

Proof. (i) $\mathbb{C}_{\kappa^+}$ is clearly κ-closed and satisfies the κ^+-c.c. by a standard Δ-system argument. Since both of these properties are preserved under finite iterations, the claim follows. To check (ii), note that any family of clubs of cardinality $<\lambda$ in a $\mathbb{Q}$-generic extension already appears at a stage before λ, by regularity of λ and Fact 1.4, and is thus pseudo-intersected by the subsequent Mathias-real. Finally, $\Vdash_{\mathbb{Q}} 2^\kappa \leq \lambda$ follows by counting nice names. □

[2] I.e., conditions in $\mathbb{C}$ are partial functions $p : \kappa \to 2$ with $|\mathrm{dom}(p)| < \kappa$, ordered by reverse inclusion.

Theorem 3.2. Let κ be strongly unfoldable and such that $2^\kappa = \kappa^+$. For any regular $\lambda > \kappa^+$, there is a forcing extension in which

$$\mathfrak{s}(\kappa) = \kappa^+ < \mathfrak{b}(\kappa) = \lambda.$$

Proof. We first force with the lottery preparation of κ (cf. Theorem 3.1 and Fact 3.2) to make the strong unfoldability of κ indestructible by any κ-closed, κ^+-c.c. forcing notion. Let $\mathbf{V}$ be the resulting new ground model. Note that in $\mathbf{V}$ we still have $2^\kappa = \kappa^+$, since the lottery preparation satisfies the κ-c.c. and has size κ. Furthermore, we have $\kappa^{<\kappa} = \kappa$ since κ is in particular strongly inaccessible.

Now, let G be $\mathbb{Q}$-generic over $\mathbf{V}$. By Lemma 3.1, κ is still strongly unfoldable in $\mathbf{V}[G]$ and therefore, by Theorem 2.5, $\mathbf{V}[G] \models \mathfrak{s}(\kappa) \geq \kappa^+$. Furthermore, $\mathbf{V}[G] \models \mathfrak{b}(\kappa) = \mathfrak{t}_{\mathrm{cl}}(\kappa) = \lambda$ by Lemma 3.1 (ii). It remains to find a splitting family of cardinality κ^+ in $\mathbf{V}[G]$.

Decompose G as $G = \langle y_\alpha : \alpha \in \kappa^+ \rangle * H$, where $\langle y_\alpha : \alpha \in \kappa^+ \rangle$ is a $\mathbb{C}_{\kappa^+}$-generic sequence of κ-reals over $\mathbf{V}$ and H is $\mathbb{M}(\mathcal{C})_\lambda$-generic over $\mathbf{V}[\langle y_\alpha : \alpha \in \kappa^+ \rangle]$. We show that the set of κ-Cohen reals $\{y_\alpha : \alpha \in \kappa^+\}$ is a splitting family in $\mathbf{V}[G]$. In $\mathbf{V}[\langle y_\alpha : \alpha \in \kappa^+ \rangle]$, let $\dot{x}$ be a nice $\mathbb{M}(\mathcal{C})_\lambda$-name for a κ-real.

Claim 3.1. There exists $\gamma \in \kappa^+$ such that $\dot{x} \in \mathbf{V}[\langle y_\alpha : \alpha \in \gamma \rangle]$.

Proof. By induction on λ. Assume the claim holds for nice $\mathbb{M}(\mathcal{C})_\alpha$-names for every $\alpha \in \lambda$. By definition, $\dot{x}$ is of the form $\dot{x} = \bigcup_{\xi \in \kappa} \{\check{\xi}\} \times A_\xi$, where A_ξ is an antichain and for each $\langle \bar{a}, \bar{C} \rangle \in A_\xi$ and each $\zeta \in \mathrm{dom}(\bar{C})$, $\bar{C}(\zeta)$ is a nice $\mathbb{M}(\mathcal{C})_\zeta$-name. Since the A_ξ have size $\leq \kappa$ by the κ^+-c.c. and since for each $\langle \bar{a}, \bar{C} \rangle \in A_\xi$: $|\mathrm{dom}(\bar{C})| < \kappa$, we have that

$$X = \bigcup_{\substack{\xi \in \kappa \\ \langle \bar{a}, \bar{C} \rangle \in A_\xi}} \mathrm{range}(\bar{C})$$

has size at most κ. By the induction hypothesis and since κ^+ is regular, there is a $\delta \in \kappa^+$ such that every $\dot{C} \in X$ already exists in $\mathbf{V}[\langle y_\alpha : \alpha \in \delta \rangle]$. Decompose $\mathbb{C}_{\kappa^+}$ as $\mathbb{C}_\delta \times \mathbb{C}_{\kappa^+ \setminus \delta}$. Since $\mathbb{C}_{\kappa^+ \setminus \delta}$ is κ-closed, it adds no new elements to $[\kappa]^{<\kappa}$ and to ${}^{<\kappa}X$, and hence, every element of $\dot{x}$ already exists in $\mathbf{V}[\langle y_\alpha : \alpha \in \delta \rangle]$. Finally, since $|\dot{x}| \leq \kappa$ and since $\mathbb{C}_{\kappa^+ \setminus \delta}$ satisfies the κ^+-c.c., there is an $S \in \mathbf{V}[\langle y_\alpha : \alpha \in \delta \rangle]$ with $|S| \leq \kappa$ such that $\dot{x} \subseteq S$. Fact 1.4 yields $\dot{x} \in \mathbf{V}[\langle y_\alpha : \alpha \in \gamma \rangle]$ for some $\delta < \gamma < \lambda$. $\vdash_{\text{Claim}}$

By the claim, $\dot{x} \in \mathbf{V}[\langle y_\alpha : \alpha \in \kappa^+ \setminus \{\gamma\} \rangle]$. We show that y_γ splits the κ-real named by $\dot{x}$. Let $\mathbf{V} := \mathbf{V}[\langle y_\alpha : \alpha \in \kappa^+ \setminus \{\gamma\} \rangle]$ be the new ground model over which we force with $\mathbb{C} * \mathbb{M}(\mathcal{C})_\lambda$. To avoid confusion, we denote $\mathbb{C}$-names with accents below the character, i.e., we let $\underset{\cdot}{y}$ be the canonical $\mathbb{C}$-name for a κ-Cohen generic real. We denote by $\underset{\sim}{\dot{x}}$ the canonical $\mathbb{C}$-name for $\dot{x} \in \mathbf{V}$.

Assume by contradiction that there exists a $\mathbb{C}$-condition p, a $\mathbb{C}$-name $\underset{\cdot}{\dot{q}}$ for an $\mathbb{M}(\mathcal{C})_\lambda$-condition and some $\eta \in \kappa$ such that either $p \Vdash_{\mathbb{C}}$ "$\underset{\cdot}{\dot{q}} \Vdash_{\mathbb{M}(\mathcal{C})_\lambda} \underset{\sim}{\dot{x}} \setminus \eta \subseteq \underset{\cdot}{y}$" or $p \Vdash_{\mathbb{C}}$ "$\underset{\cdot}{\dot{q}} \Vdash_{\mathbb{M}(\mathcal{C})_\lambda} \underset{\sim}{\dot{x}} \cap \underset{\cdot}{y} \subseteq \eta$". Since $\mathbb{C}$ is κ-closed, we can assume without loss of generality that there is some $\bar{a} \in \mathbf{V}$ such that $\underset{\cdot}{\dot{q}} = \mathrm{op}(\bar{a}, \underset{\cdot}{\bar{C}})$, where $\underset{\cdot}{\bar{C}}$ is a $\mathbb{C}$-name for a sequence of names for clubs.

Let y be a κ-Cohen real over $\mathbf{V}$ such that $p \subseteq y$, i.e., such that p is in the generic filter. Define the $\mathbb{C}$-generic κ-real y' as follows: For $\zeta \in \mathrm{dom}(p)$, $y'(\zeta) := y(\zeta) = p(\zeta)$ and else $y'(\zeta) := 1 - y(\zeta)$. Note that $\mathbf{V}[y'] = \mathbf{V}[y] =: \mathbf{W}$.

In $\mathbf{W}$, we are in one of the following two cases:

(i) Either $\langle \bar{a}, \underset{\cdot}{\bar{C}}[y] \rangle \Vdash$ "$\dot{x} \setminus \eta \subseteq y$" and $\langle \bar{a}, \underset{\cdot}{\bar{C}}[y'] \rangle \Vdash$ "$\dot{x} \setminus \eta \subseteq y'$", or

(ii) $\langle \bar{a}, \underset{\cdot}{\bar{C}}[y] \rangle \Vdash$ "$\dot{x} \cap y \subseteq \eta$" and $\langle \bar{a}, \underset{\cdot}{\bar{C}}[y'] \rangle \Vdash$ "$\dot{x} \cap y' \subseteq \eta$".

Since the two $\mathbb{M}(\mathcal{C})_\lambda$-conditions $\langle \bar{a}, \underset{\cdot}{\bar{C}}[y] \rangle$ and $\langle \bar{a}, \underset{\cdot}{\bar{C}}[y'] \rangle$ are compatible by Fact 3.4 (ii) with common extension r, either $r \Vdash$ "$\dot{x} \setminus \eta \subseteq y$ and $\dot{x} \setminus \eta \subseteq y'$" if we are in Case (i), or $r \Vdash$ "$\dot{x} \cap y \subseteq \eta$ and $\dot{x} \cap y' \subseteq \eta$" if we are in Case (ii). In Case (i), $r \Vdash$ "$\dot{x} \setminus \eta \subseteq y \cap y'$" and thus $r \Vdash$ "$|\dot{x}| < \kappa$". In Case (ii), $r \Vdash$ "$\dot{x} \cap (y \cup y') \subseteq \eta$" and thus also $r \Vdash$ "$|\dot{x}| < \kappa$". □

3.2 $\mathfrak{s}(\kappa) = \kappa^+ < \mathfrak{b}(\kappa)$ at Many κ Simultaneously

While the global results in the next chapter will deal with *all* regular κ simultaneously, which necessitates the use of a product forcing, the situation here is somewhat more relaxed: We are only dealing with strongly unfoldable cardinals, and these are 'spread out', in the sense that the stage of the forcing construction dealing with some strongly unfoldable κ is small enough to not interfere with the situation at larger strongly unfoldable cardinals: Note that if $\kappa < \kappa'$ are strongly unfoldable, we cannot force $\mathfrak{b}(\kappa) \geq \kappa'$ without destroying even the strong inaccessibility of κ'. The following lemma shows that the strong unfoldability of κ' is preserved if we act responsibly and force $\mathfrak{b}(\kappa) < \kappa'$. It is very likely folklore.[3]

Lemma 3.2. Let κ be a strongly unfoldable cardinal and $\mathbb{Q} \in \mathcal{H}_\kappa$ a forcing notion. Then, κ remains strongly unfoldable in any $\mathbb{Q}$-generic extension.

[3]A sketch of the argument can be found in [26, pp. 1227-1228].

Proof. This essentially follows from a simplified version of the proof of Theorem 3.1. Let θ be any ordinal and G a $\mathbb{Q}$-generic filter over the ground model $\mathbf{V}$. We show that κ is θ-strongly unfoldable in $\mathbf{V}[G]$, using Fact 3.3. Thus, let A be a subset of κ in $\mathbf{V}[G]$ that we need to put into a κ-model with a θ-strong unfoldability embedding.

Let $\dot{A}$ be a nice $\mathbb{Q}$-name for A and note that $|\text{trcl}(\dot{A})| = \kappa$. Choose any regular $\lambda > \kappa$ and let $X \prec \mathcal{H}_\lambda$ be an elementary submodel such that $\mathcal{H}_\kappa \cup \text{trcl}(\dot{A}) \cup \{\kappa\} \subseteq X$, $|X| = \kappa$ and so that ${}^{<\kappa}X \subseteq X$. Note that this is possible because $|\mathcal{H}_\kappa| = 2^{<\kappa} = \kappa$. Let $\pi : X \to M$ be the Mostowski collapse of X. Note that M is a κ-model and that $\pi|_{\mathcal{H}_\kappa \cup \{\dot{A}\}} = \text{id}$, giving $\mathbb{Q}, \dot{A} \in M$. We can thus directly force with $\mathbb{Q}$ over M using G to obtain the κ-model $M[G]$ containing A.

Fix some θ-strong unfoldability embedding $j : M \to N$. Since j has critical point κ and N contains $\mathbf{V}_\theta$, $j|_{\mathcal{H}_\kappa} = \text{id}$ and thus $\mathbb{Q} \in N$. The embedding $j : M \to N$ therefore trivially lifts to $j : M[G] \to N[G]$ and since $G \in N[G]$: $\mathbf{V}[G]_\theta \subseteq \mathbf{V}_\theta[G] \subseteq N[G]$. □

As stated before, we would like to control cardinal characteristics at *all* regular κ simultaneously, not just at a *set* of regular κ. While the simple method of globally separating $\mathfrak{s}(\kappa)$ and $\mathfrak{b}(\kappa)$ presented below does not quite work for all strongly unfoldable κ simultaneously – strongly unfoldable limits of strongly unfoldable cardinals need to be left out – we still want to treat a proper class of κ. This requires the use of class forcing, i.e., forcing with a partial order that is a proper class. Class forcing is generally tricky since ZFC may not hold in generic extensions. However, the class forcing we use below, as well as the class forcings we encounter in the next chapter, behave very nicely in that regard. We will first prove the main result of this chapter, modulo the class forcing complication, and then explain briefly why treating a proper class instead of a set is unproblematic.

Definition 3.6. A function E is an *index function* if $\text{dom}(E)$ is a class of regular cardinals. If E is an index function and $\kappa \in \text{dom}(E)$, we let $E^{<\kappa} := E|_\kappa$, $E^{\leq\kappa} := E|_{\kappa+1}$ and $E^{>\kappa} := E|_{\text{dom}(E)\setminus(\kappa+1)}$.

Theorem 3.3. Assume $\mathbf{V} \models \mathsf{GCH}$. Let E be an index function defined on strongly unfoldable cardinals such that

(i) For every $\kappa \in \text{dom}(E) : \kappa^+ < E(\kappa)$.

(ii) For every $\kappa \in \text{dom}(E) : \bigcup\{E(\delta) : \delta \in \text{dom}(E^{<\kappa})\} < \kappa$.

Then, there is a class forcing extension of $\mathbf{V}$ in which

$$\forall\kappa \in \text{dom}(E) : \mathfrak{s}(\kappa) = \kappa^+ < \mathfrak{b}(\kappa) = E(\kappa).$$

Proof. For every $\kappa \in \mathrm{dom}(E)$, let $\mu_\kappa := \bigcup\{E(\delta) : \delta \in \mathrm{dom}(E^{<\kappa})\} < \kappa$ and denote by $\mathbb{Q}_\kappa = \mathbb{C}_{\kappa^+} * \dot{\mathbb{M}}(\mathcal{C})_{E(\kappa)}$ the forcing notion developed in Theorem 3.2. Fix for every $\kappa \in \mathrm{dom}(E)$ a Menas function f'_κ for κ and let $f_\kappa := f'_\kappa|_{(\mu_\kappa^+,\, \kappa)}$, which clearly still has the Menas property for κ. Let $\mathbb{P}_\kappa$ be the lottery preparation of κ relative to f_κ and note that $\mathbb{P}_\kappa$ is μ_κ^+-closed, because all of its stages up to stage μ_κ^+ are trivial.

Define $\mathbb{R}_\kappa := \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$. Note that if $\kappa < \kappa'$ are in $\mathrm{dom}(E)$, then $\mathbb{R}_\kappa \in \mathcal{H}_{\kappa'}$, by the strong inaccessibility of κ'. Furthermore, $\mathbb{R}_\kappa$ is μ_κ^+-closed and satisfies the κ^+-c.c..

Let $\mathbb{E}$ be the Easton support iteration over $\mathrm{dom}(E)$ of the $\mathbb{R}_\kappa$, i.e., at stage α we force with $\mathbb{R}_\alpha$ iff $\alpha \in \mathrm{dom}(E)$ and with the trivial forcing notion otherwise. Now, for any $\kappa \in \mathrm{dom}(E)$, $\mathbb{E}$ is forcing equivalent to the three step iteration $\mathbb{E}^{<\kappa} * \dot{\mathbb{R}}_\kappa * \dot{\mathbb{E}}^{\mathrm{tail}}$, where $\mathbb{E}^{<\kappa} = \mathbb{E}|_\kappa$ is in $\mathcal{H}_\kappa$, $\dot{\mathbb{R}}_\kappa$ is an $\mathbb{E}^{<\kappa}$-name for $\mathbb{R}_\kappa$ in the $\mathbb{E}^{<\kappa}$-generic extension and $\dot{\mathbb{E}}^{\mathrm{tail}}$ is a $(\mathbb{E}^{<\kappa} * \dot{\mathbb{R}}_\kappa)$-name such that $\Vdash_{\mathbb{E}^{<\kappa} * \dot{\mathbb{R}}_\kappa}$ "$\dot{\mathbb{E}}^{\mathrm{tail}}$ is the Easton support iteration of the $\mathbb{R}_\kappa$ over $\mathrm{dom}(E^{>\kappa})$". Since $\dot{\mathbb{E}}^{\mathrm{tail}}$ only forces with necessarily $E(\kappa)^+$-closed forcing notions, we have $\Vdash_{\mathbb{E}^{<\kappa} * \dot{\mathbb{R}}_\kappa}$ "$\dot{\mathbb{E}}^{\mathrm{tail}}$ is $E(\kappa)^+$-closed" (cf. [10, Proposition 7.12]).

Let G be some $\mathbb{E}$-generic filter and split G as $G^{<\kappa} * H * G^{\mathrm{tail}}$. Since $\mathbb{E}^{<\kappa} \in \mathcal{H}_\kappa$, it follows from Lemma 3.2 that κ remains strongly unfoldable in $\mathbf{V}[G^{<\kappa}]$. The equality $2^\kappa = \kappa^+$ is clearly also preserved. Thus, by Theorem 3.2, $\mathbf{V}[G^{<\kappa} * H] \models \mathfrak{s}(\kappa) = \kappa^+ < \mathfrak{b}(\kappa) = E(\kappa) = 2^\kappa$. Since $\dot{\mathbb{E}}^{\mathrm{tail}}[G^{<\kappa} * H]$ is $E(\kappa)^+$-closed, this is preserved in $\mathbf{V}[G]$. $\square$

The reason this works in case $\mathrm{dom}(E)$ is a proper class instead of a set, is that $\mathbb{E}$ is a *progressively closed* iteration: Denoting by Ω the class of ordinals, an ordinal length product $\Pi_{\xi\in\Omega}\mathbb{Q}_\xi$ or an ordinal length iteration $\langle\langle\mathbb{P}_\xi : \xi \in \Omega\rangle, \langle\dot{\mathbb{Q}}_\xi : \xi \in \Omega\rangle\rangle$ is *progressively closed* iff for every ordinal α, there is some ordinal β such that for all $\xi \geq \beta$, $\mathbb{Q}_\xi$ is α-closed (in the case of products), or $\Vdash_{\mathbb{P}_\xi}$ "$\dot{\mathbb{Q}}_\xi$ is α-closed" (in the case of iterations). Intuitively, such products and iterations behave nicely because every initial segment of the universe is only affected by a set-sized segment of the product/iteration. For the details, see [34].

Chapter 4

Generalized Tower Spectra

While the content of the previous chapters has, to some extent, already intersected with the question of whether towers of certain heights exist, that question will be the focus of this final chapter. The results presented here also constitute the main focus of my thesis work and will appear in the Journal of Symbolic Logic. Apart from Theorem 4.4, they deal with $\mathfrak{sp}(\mathfrak{t}(\kappa))$ at all regular κ simultaneously. As explained on the previous page, we can ignore the resulting class forcing complication since all the forcing notions used will be progressively closed Easton products.

4.1 Globally Small Tower Spectra

4.1.1 In the Easton Model

We begin by showing that $\mathfrak{sp}(\mathfrak{t}(\kappa)) = \{\kappa^+\}$ is consistent globally with an arbitrarily large generalized continuum 2^κ. In fact, this holds in the Easton model, where the class function $\kappa \to 2^\kappa$ can realize any pattern permitted by König's Theorem. The argument used here – an isomorphism of names – serves as an easy blueprint for the more involved isomorphism-of-names-arguments required in the later results.

Definition 4.1. Recall that E is an index function iff $\mathrm{dom}(E)$ is a class of regular cardinals (cf. Definition 3.6). An index function E is an *Easton function* if for every $\kappa \in \mathrm{dom}(E)$, $E(\kappa)$ is a cardinal with $\mathrm{cf}(E(\kappa)) > \kappa$ and such that $\kappa < \kappa' \implies E(\kappa) \leq E(\kappa')$.

If there is a forcing notion $\mathbb{P}_\kappa$ for each $\kappa \in \mathrm{dom}(E)$, we can define the Easton product $\mathbb{P}(E)$ of the $\mathbb{P}_\kappa$ as the product with Easton support over $\mathrm{dom}(E)$ of the $\mathbb{P}_\kappa$.

Thus, $\mathbb{P}(E)$ consists of conditions of the form $p = \langle p(\kappa) : \kappa \in \mathrm{dom}(E)\rangle$, where for each regular cardinal $\gamma : |\{\kappa \in \mathrm{dom}(E) : p(\kappa) \neq \mathbb{1}\} \cap \gamma| < \gamma$. Recall that the support of p is the set $\mathrm{supp}(p) = \{\kappa \in \mathrm{dom}(E) : p(\kappa) \neq \mathbb{1}\}$. It is clear that $\mathbb{P}(E)$ is isomorphic to $\mathbb{P}(E^{\leq\kappa}) \times \mathbb{P}(E^{>\kappa})$.

Definition 4.2. Let E be an Easton function. *Easton forcing* relative to E is the Easton product of the forcing notions $\mathrm{Fn}_{<\kappa}(E(\kappa) \times \kappa, 2)$ over all $\kappa \in \mathrm{dom}(E)$.

It is well-known and easy to check that for each $\kappa \in \mathrm{dom}(E) : \mathbb{P}(E^{\leq\kappa})$ satisfies the κ^+-c.c. and $\mathbb{P}(E^{>\kappa})$ is κ^+-closed, provided that $2^{<\kappa} = \kappa$.

Theorem 4.1. Let $\mathbf{V} \models \mathsf{GCH}$, let E be an Easton function and denote Easton forcing relative to E by $\mathbb{P}(E)$. Then, in any $\mathbb{P}(E)$-generic extension of $\mathbf{V}$:

$$\forall \kappa \in \mathrm{dom}(E) : \mathfrak{sp}(\mathfrak{t}(\kappa)) = \{\kappa^+\} \text{ and } 2^\kappa = E(\kappa).$$

Proof. The second equality is well-known. Fix $\kappa \in \mathrm{dom}(E)$ and let G be $\mathbb{P}(E)$ generic over $\mathbf{V}$. Assume by contradiction that there exists a κ-tower $\langle a_\xi : \xi \in \lambda\rangle$ of length $\lambda \geq \kappa^{++}$ in $\mathbf{V}[G]$. We can assume that $\langle a_\xi : \xi \in \lambda\rangle$ is strictly $\supseteq^*$-descending, by extracting such a subsequence. Decompose $\mathbb{P}(E)$ as $\mathbb{P}(E^{\leq\kappa}) \times \mathbb{P}(E^{>\kappa})$ and $G = G^{\leq\kappa} \times G^{>\kappa}$ accordingly. Since $\mathbb{P}(E^{>\kappa})$ is κ^+-closed, the GCH at $\delta \leq \kappa$ still holds in $\mathbf{V}[G^{>\kappa}]$ and $(\mathbb{P}(E^{\leq\kappa}))^{\mathbf{V}[G^{>\kappa}]} = (\mathbb{P}(E^{\leq\kappa}))^{\mathbf{V}}$. We designate $\mathbf{V}[G^{>\kappa}]$ as the new ground model.

For each $\xi \in \kappa^{++}$, let $\dot{a}_\xi$ be a nice $\mathbb{P}(E^{\leq\kappa})$-name for a_ξ and let $p_0 \in G^{\leq\kappa}$ be a $\mathbb{P}(E^{\leq\kappa})$-condition such that $\forall \xi < \xi' < \kappa^{++} : p_0 \Vdash_{\mathbb{P}(E^{\leq\kappa})}$ "$\dot{a}_\xi \supsetneq^* \dot{a}_{\xi'}$".

Any nice $\mathbb{P}(E^{\leq\kappa})$-name $\dot{x}$ is of the form $\dot{x} = \bigcup_{\alpha\in\kappa} \{\check{\alpha}\} \times A_\alpha(\dot{x})$, where $A_\alpha(\dot{x})$ is an antichain in $\mathbb{P}(E^{\leq\kappa})$. Since $\mathbb{P}(E^{\leq\kappa})$ satisfies the κ^+-c.c., the set

$$S^\delta(\dot{x}) := \bigcup_{\substack{\alpha\in\kappa \\ p\in A_\alpha(\dot{x})}} \mathrm{dom}(p(\delta)) \cup \mathrm{dom}(p_0(\delta))$$

has cardinality at most κ for every $\delta \in \mathrm{dom}(E^{\leq\kappa})$, and thus the same holds for the set $S(\dot{x}) := \bigcup_{\delta\in\mathrm{dom}(E^{\leq\kappa})} S^\delta(\dot{x})$.

By applying the Δ-system Lemma, which requires the GCH at κ, to the family $\{S(\dot{a}_\xi) : \xi \in \kappa^{++}\}$, we find some $X \subseteq \kappa^{++}$ of cardinality κ^{++} and a sequence $\langle R^\delta : \delta \in \mathrm{dom}(E^{\leq\kappa})\rangle$ such that for all $\xi \neq \xi' \in X$ and all $\delta \in \mathrm{dom}(E^{\leq\kappa}) : S^\delta(\dot{a}_\xi) \cap S^\delta(\dot{a}_{\xi'}) = R^\delta$. Note that $\mathrm{dom}(p_0(\delta)) \subseteq R^\delta$. Since $S^\delta(\dot{a}_\xi)$ has cardinality $\leq\kappa$ and since $\kappa^\kappa = \kappa^+$, we find by the pigeonhole principle some $X' \subseteq X$ of cardinality κ^{++} such that $|S^\delta(\dot{a}_\xi) \setminus R^\delta| = |S^\delta(\dot{a}_{\xi'}) \setminus R^\delta|$ for all $\xi \neq \xi' \in X'$ and $\delta \in \mathrm{dom}(E^{\leq\kappa})$.

Fix some $\xi_0 \in X'$ and choose for each $\xi \in X'$ and each $\delta \in \operatorname{dom}(E^{\leq\kappa})$ a permutation of $E(\delta) \times \delta$ of order 2 that maps $S^\delta(\dot{a}_\xi)$ to $S^\delta(\dot{a}_{\xi_0})$ and fixes everything besides $S^\delta(\dot{a}_\xi) \cup S^\delta(\dot{a}_{\xi_0}) \setminus R^\delta$. Denote by φ^δ_ξ the automorphism of $\mathrm{Fn}_{<\delta}(E(\delta) \times \delta, 2)$ that this permutation induces. By applying these automorphisms coordinate-wise, we obtain automorphisms of $\mathbb{P}(E^{\leq\kappa})$, which we denote by φ_ξ. Since we chose permutations fixing the R^δ, we have $\varphi_\xi(p_0) = p_0$. The automorphisms φ_ξ extend to $\mathbb{P}(E^{\leq\kappa})$-names in the obvious way.

Note that $\varphi_\xi(\dot{a}_\xi)$ is a nice name with $S(\varphi_\xi(\dot{a}_\xi)) \subseteq S(\dot{a}_{\xi_0})$. By counting, we see that there are at most κ^+ many nice names $\dot{x}$ with $S(\dot{x}) \subseteq S(\dot{a}_{\xi_0})$. Therefore, there exists $X'' \subseteq X'$ of cardinality κ^{++} and a nice name $\dot{x}$ such that $\varphi_\xi(\dot{a}_\xi) = \dot{x}$ for every $\xi \in X''$.

Now, fix $\xi < \xi' \in X'' \setminus \{\xi_0\}$ and define the following automorphism of $\mathbb{P}(E^{\leq\kappa})$:

$$\chi := \varphi_\xi \circ \varphi_{\xi'} \circ \varphi_\xi.$$

Note that $\chi(\dot{a}_\xi) = \dot{a}_{\xi'}$, that $\chi(\dot{a}_{\xi'}) = \dot{a}_\xi$ and that $\chi(p_0) = p_0$. By assumption, $p_0 \Vdash_{\mathbb{P}(E^{\leq\kappa})} \dot{a}_\xi \supsetneq^* \dot{a}_{\xi'}$. Thus, $\chi(p_0) \Vdash_{\chi(\mathbb{P}(E^{\leq\kappa}))} \chi(\dot{a}_\xi) \supsetneq^* \chi(\dot{a}_{\xi'})$, which implies that

$$p_0 \Vdash \dot{a}_{\xi'} \supsetneq^* \dot{a}_\xi \text{ and } \dot{a}_\xi \supsetneq^* \dot{a}_{\xi'},$$

a contradiction. □

4.1.2 With Arbitrarily Large MAD Spectra

The above result can be generalized to show that consistently, the κ-tower spectrum equals $\{\kappa^+\}$ for all regular κ, while the κ-MAD spectrum is arbitrarily large. More precisely, we prove the following:

Theorem 4.2. Let $\mathbf{V} \models \mathsf{GCH}$ and let E be an index function such that for every $\kappa \in \operatorname{dom}(E)$, $E(\kappa)$ is a closed set of cardinals with $\min E(\kappa) \geq \kappa^+$, $\operatorname{cf}(\max E(\kappa)) > \kappa$ and such that $\kappa < \kappa' \implies \max E(\kappa) \leq \max E(\kappa')$. There is a forcing extension of $\mathbf{V}$ in which

$$\forall \kappa \in \operatorname{dom}(E) : \mathfrak{sp}(\mathfrak{t}(\kappa)) = \{\kappa^+\},\ E(\kappa) \subseteq \mathfrak{sp}(\mathfrak{a}(\kappa)) \text{ and } 2^\kappa = \max E(\kappa).$$

This is based on a construction by Bağ, Fischer and Friedman [2]. As is shown in that paper, the same construction allows for more accurate control of $\mathfrak{sp}(\mathfrak{a}(\kappa))$ by restricting the domain of E to successors of regular cardinals together with $\aleph_0$, and the range of E to so-called κ-Blass spectra. While the definition of a κ-Blass spectrum is not necessary for our purposes, we give it for the sake of completeness.

Definition 4.3 ([2], Definition 2.1)**.** A *κ-Blass spectrum* is a set A of cardinals satisfying $\min A = \kappa^+$, $\forall \mu \in A : [\mathrm{cf}(\mu) \leq \kappa \implies \mu^+ \in A]$ and $\gamma \in A$ for every cardinal $\kappa^+ \leq \gamma \leq |A|$.

Corollary 4.1 (GCH)**.** If E is defined on successors of regular cardinals together with $\aleph_0$, and $E(\kappa)$ is a κ-Blass spectrum for every $\kappa \in \mathrm{dom}(E)$, we consistently have

$$\forall \kappa \in \mathrm{dom}(E) : \mathfrak{sp}(\mathfrak{t}(\kappa)) = \{\kappa^+\} \text{ and } \mathfrak{sp}(\mathfrak{a}(\kappa)) = E(\kappa).$$

Proof of Theorem 4.2. We begin by defining the relevant forcing notion. It is essentially a global version of MAD-Hechler forcing introduced in Definition 2.5.

Definition 4.4 ([2], Definition 4.2)**.** Define for each $\kappa \in \mathrm{dom}(E)$ and each $\lambda \in E(\kappa)$ the following forcing notion $\mathbb{A}^{\kappa,\lambda}$: An $\mathbb{A}^{\kappa,\lambda}$-condition is a function $p : \Delta^p \to [\kappa]^{<\kappa}$, where $\Delta^p \in [\lambda]^{<\kappa}$. We define $p' \leq p$ iff

(i) $\Delta^p \subseteq \Delta^{p'}$,

(ii) $\forall x \in \Delta^p : p(x) \subseteq p'(x)$,

(iii) $\forall \eta_1 \neq \eta_2 \in \Delta^p : p'(\eta_1) \cap p'(\eta_2) \subseteq p(\eta_1) \cap p(\eta_2)$.

For each $\kappa \in \mathrm{dom}(E)$, let $\mathbb{A}^\kappa$ be the $<\kappa$-support product of the $\mathbb{A}^{\kappa,\lambda}$ over all $\lambda \in E(\kappa)$. Then, let $\mathbb{A}$ be the Easton product of the $\mathbb{A}^\kappa$.

Let G be $\mathbb{A}$-generic over $\mathbf{V}$. It is shown in [2, Theorem 4.6 and Remark 4.7] that for all $\kappa \in \mathrm{dom}(E) : E(\kappa) \subseteq \mathfrak{sp}(\mathfrak{a}(\kappa))$ and $2^\kappa = \max E(\kappa)$ holds in $\mathbf{V}[G]$. To show the other equality, let $\kappa \in \mathrm{dom}(E)$ and decompose $\mathbb{A}$ as $\mathbb{A}^{>\kappa} \times \mathbb{A}^{\leq\kappa}$ and $G = G^{>\kappa} \times G^{\leq\kappa}$ accordingly. As is shown in [2, Lemma 4.3], $\mathbb{A}^{>\kappa}$ is κ^+-closed and $\mathbb{A}^{\leq\kappa}$ satisfies the κ^+-c.c., which implies that the GCH at $\delta \leq \kappa$ still holds in $\mathbf{V}[G^{>\kappa}]$ and that $(\mathbb{A}^{\leq\kappa})^{\mathbf{V}[G^{>\kappa}]} = (\mathbb{A}^{\leq\kappa})^{\mathbf{V}}$. Let $\mathbf{W} := \mathbf{V}[G^{>\kappa}]$ be the new ground model.

Assume by contradiction that $\langle a_\xi : \xi < \kappa^{++} \rangle$ is a strictly $\supseteq^*$-descending sequence of cofinal subsets of κ in $\mathbf{W}[G^{\leq\kappa}]$. Let $\dot{a}_\xi$ be a nice $\mathbb{A}^{\leq\kappa}$-name for a_ξ and let p_0 be such that for all $\xi < \xi' < \kappa^{++} : p_0 \Vdash \dot{a}_\xi \supsetneq^* \dot{a}_{\xi'}$.

In order to find the required isomorphisms, we must first extend the forcing notion $\mathbb{A}^{\leq\kappa}$ to a larger forcing notion $\mathbb{Q}^{\leq\kappa}$ into which $\mathbb{A}^{\leq\kappa}$ completely embeds.

Definition 4.5. For every $\delta \in \mathrm{dom}(E^{\leq\kappa})$, let $b^\delta := |E(\delta)|$ and $J^\delta := \max E(\delta)$, and for every $\beta \in b^\delta$, let $\mathbb{Q}^{\delta,\beta}$ be the forcing notion $\mathbb{A}^{\delta,J_\delta}$. Let $\mathbb{Q}^\delta$ be the $<\kappa$-support product of the $\mathbb{Q}^{\delta,\beta}$ and $\mathbb{Q}^{\leq\kappa}$ the Easton product over all $\delta \in \mathrm{dom}(E^{\leq\kappa})$ of the $\mathbb{Q}^\delta$.

It is easy to verify that $\mathbb{A}^{\leq\kappa}$ completely embeds into $\mathbb{Q}^{\leq\kappa}$ (see [2, Lemma 4.8]). Thus, $\forall \xi < \xi' < \kappa^{++} : p_0 \Vdash_{\mathbb{Q}^{\leq\kappa}} \dot{a}_\xi \supsetneq^* \dot{a}_{\xi'}$.

Definition 4.6. Let $\dot{x}$ be a nice $\mathbb{Q}^{\leq\kappa}$-name for a subset of κ, i.e., $\dot{x} = \bigcup_{\alpha\in\kappa}\{\check{\alpha}\} \times A_\alpha(\dot{x})$. For each $\delta \in \mathrm{dom}(E^{\leq\kappa})$ and $\beta \in b^\delta$, define the following sets:

$$\mathrm{supp}^\delta(\dot{x}) := \bigcup_{\substack{\alpha\in\kappa \\ p\in A_\alpha(\dot{x})}} \mathrm{supp}(p(\delta)) \cup \mathrm{supp}(p_0(\delta)) \in [b^\delta]^{\leq\kappa}$$

$$\Delta^{\delta,\beta}(\dot{x}) := \bigcup_{\substack{\alpha\in\kappa \\ p\in A_\alpha(\dot{x})}} \Delta^{p(\delta)(\beta)} \cup \Delta^{p_0(\delta)(\beta)} \in [J^\delta]^{\leq\kappa}$$

By applying the Δ-system Lemma, we obtain some $X \subseteq \kappa^{++}$ of cardinality κ^{++} and for each $\delta \in \mathrm{dom}(E^{\leq\kappa})$ a root R^δ such that for all $\xi \neq \xi' \in X$: $\mathrm{supp}^\delta(\dot{a}_\xi) \cap \mathrm{supp}^\delta(\dot{a}_{\xi'}) = R^\delta$. Since $\kappa^\kappa = \kappa^+ < \kappa^{++}$, we can assume without loss of generality that for every $\delta \in \mathrm{dom}(E^{\leq\kappa})$, the value $|\mathrm{supp}^\delta(\dot{a}_\xi) \setminus R^\delta|$ does not depend on $\xi \in X$.

Fix some $\xi_0 \in X$ and let ψ^δ_ξ be a permutation of b^δ of order 2 that maps $\mathrm{supp}^\delta(\dot{a}_\xi)$ to $\mathrm{supp}^\delta(\dot{a}_{\xi_0})$ and fixes everything outside of $(\mathrm{supp}^\delta(\dot{a}_\xi) \cup \mathrm{supp}^\delta(\dot{a}_{\xi_0})) \setminus R^\delta$. This permutation naturally induces an automorphisms of $\mathbb{Q}^\delta$. By applying these automorphisms coordinate-wise, we obtain for each $\xi \in X$ an automorphism of the entire $\mathbb{Q}^{\leq\kappa}$, which we call ψ_ξ. It recursively extends to $\mathbb{Q}^{\leq\kappa}$-names. Note that $\forall\delta \in \mathrm{dom}(E^{\leq\kappa})$: $\mathrm{supp}^\delta(\psi_\xi(\dot{a}_\xi)) = \mathrm{supp}^\delta(\dot{a}_{\xi_0})$, $\psi_\xi(p_0) = p_0$ and for every $\xi' \in X \setminus \{\xi, \xi_0\} : \psi_\xi(\dot{a}_{\xi'}) = \dot{a}_{\xi'}$.

In an abuse of notation, we assume that the sets J^δ underlying the forcing notions $\mathbb{Q}^{\delta,\beta}$ are disjoint for different (δ, β) and apply the Δ-system Lemma to the family

$$\left\{\bigcup\{\Delta^{\delta,\beta}(\psi_\xi(\dot{a}_\xi)) :\ \delta \in \mathrm{dom}(E^{\leq\kappa}),\ \beta \in b^\delta\} : \xi \in X\right\}.$$

We obtain some $X' \subseteq X$ of cardinality κ^{++} and for each $\delta \in \mathrm{dom}(E^{\leq\kappa})$ and each $\beta \in b^\delta$ a root $R^{\delta,\beta}$, i.e., we have for all $\xi \neq \xi' \in X'$, every $\delta \in \mathrm{dom}(E^{\leq\kappa})$ and every $\beta \in b^\delta$: $\Delta^{\delta,\beta}(\psi_\xi(\dot{a}_\xi)) \cap \Delta^{\delta,\beta}(\psi_{\xi'}(\dot{a}_{\xi'})) = R^{\delta,\beta}$.

Since $\mathrm{supp}^\delta(\psi_\xi(\dot{a}_\xi)) = \mathrm{supp}^\delta(\dot{a}_{\xi_0})$, and since $\kappa^\kappa < \kappa^{++}$, we can again assume without loss of generality that the value $|\Delta^{\delta,\beta}(\psi_\xi(\dot{a}_\xi)) \setminus R^{\delta,\beta}|$ does not depend on $\xi \in X'$. We may therefore fix $\xi_1 \in X'$ and choose for each $\delta \in \mathrm{dom}(E^{\leq\kappa})$ and $\beta \in b^\delta$ some permutation $\varphi^{\delta,\beta}_\xi$ of order 2 of J^δ that maps $\Delta^{\delta,\beta}(\psi_\xi(\dot{a}_\xi))$ to $\Delta^{\delta,\beta}(\psi_{\xi_1}(\dot{a}_{\xi_1}))$, and fixes everything except for $(\Delta^{\delta,\beta}(\psi_\xi(\dot{a}_\xi)) \cup \Delta^{\delta,\beta}(\psi_{\xi_1}(\dot{a}_{\xi_1}))) \setminus R^{\delta,\beta}$. This map induces an automorphism of $\mathbb{Q}^{\delta,\beta}$, and by applying the maps coordinate-wise, we again obtain an automorphism of the entire $\mathbb{Q}^{\leq\kappa}$, which we denote by φ_ξ. Note that $\varphi_\xi(p_0) = p_0$ and for every $\xi' \in X' \setminus \{\xi, \xi_1\} : \varphi_\xi(\dot{a}_{\xi'}) = \dot{a}_{\xi'}$.

By definition of the maps, $\varphi_\xi \circ \psi_\xi(\dot{a}_\xi)$ is a nice name satisfying for every $\delta \in$

dom($E^{\leq\kappa}$) and $\beta \in b^\delta$:

$$\mathrm{supp}^\delta(\varphi_\xi \circ \psi_\xi(\dot{a}_\xi)) = \mathrm{supp}^\delta(\dot{a}_{\xi_0}) \text{ and } \Delta^{\delta,\beta}(\varphi_\xi \circ \psi_\xi(\dot{a}_\xi)) = \Delta^{\delta,\beta}(\psi_{\xi_1}(\dot{a}_{\xi_1})).$$

By an easy counting argument, there are at most κ^+ many nice names with this property, which implies that there exist fixed $\xi \neq \xi' \in X' \setminus \{\xi_0, \xi_1\}$ and a nice name $\dot{z}$ such that $\varphi_\xi \circ \psi_\xi(\dot{a}_\xi) = \varphi_{\xi'} \circ \psi_{\xi'}(\dot{a}_{\xi'}) = \dot{z}$.

Since we have fixed ξ and ξ', we will from now on use the shorthands $\psi := \psi_\xi$, $\psi' := \psi_{\xi'}$, $\varphi := \varphi_\xi$, $\varphi' := \varphi_{\xi'}$. The rest of the proof consists in showing that the automorphism

$$\chi := \psi' \circ \varphi' \circ \psi \circ \varphi \circ \varphi' \circ \psi' \circ \varphi \circ \psi \circ \varphi \circ \psi$$

satisfies $\chi(\dot{a}_\xi) = \dot{a}_{\xi'}$ and $\chi(\dot{a}_{\xi'}) = \dot{a}_\xi$. Unfortunately, there does not seem to be a shorter one that works. Since $\chi(p_0) = p_0$, we obtain the contradiction

$$p_0 \Vdash_{\mathbb{Q}^{\leq\kappa}} \dot{a}_\xi \subsetneq^* \dot{a}_{\xi'} \wedge \dot{a}_{\xi'} \subsetneq^* \dot{a}_\xi,$$

just as in the proof of Theorem 4.1.

Definition 4.7. Let $U^\delta := \mathrm{supp}^\delta(\dot{a}_{\xi_0})$, $U^{\delta,\beta} := \Delta^{\delta,\beta}(\psi_{\xi_1}(\dot{a}_{\xi_1}))$ and define the following subsets of $\mathbb{Q}^{\leq\kappa}$:

(i) $\mathbb{R}^{\mathsf{R}} := \{p \in \mathbb{Q}^{\leq\kappa} : \forall\delta \in \mathrm{dom}(E^{\leq\kappa})\, \forall\beta \in b^\delta : \mathrm{supp}(p(\delta)) \subseteq R^\delta \wedge \Delta^{p(\delta)(\beta)} \subseteq R^{\delta,\beta}\}$

(ii) $\mathbb{R}^{\mathsf{U}} := \{p \in \mathbb{Q}^{\leq\kappa} : \forall\delta \in \mathrm{dom}(E^{\leq\kappa})\, \forall\beta \in b^\delta : \mathrm{supp}(p(\delta)) \subseteq R^\delta \wedge \Delta^{p(\delta)(\beta)} \subseteq U^{\delta,\beta} \setminus R^{\delta,\beta}\}$

(iii) $\mathbb{R}^{\xi} := \{p \in \mathbb{Q}^{\leq\kappa} : \forall\delta \in \mathrm{dom}(E^{\leq\kappa})\, \forall\beta \in b^\delta : \mathrm{supp}(p(\delta)) \subseteq R^\delta \wedge \Delta^{p(\delta)(\beta)} \subseteq \Delta^{\delta,\beta}(\dot{a}_\xi) \setminus R^{\delta,\beta}\}$, and define $\mathbb{R}^{\xi'}$ analogously.

(iv) $\mathbb{U} := \{p \in \mathbb{Q}^{\leq\kappa} : \forall\delta \in \mathrm{dom}(E^{\leq\kappa}) : \mathrm{supp}(p(\delta)) \subseteq U^\delta \setminus R^\delta\}$

(v) $\mathbb{P}^\xi := \{p \in \mathbb{Q}^{\leq\kappa} : \forall\delta \in \mathrm{dom}(E^{\leq\kappa}) : \mathrm{supp}(p(\delta)) \subseteq \mathrm{supp}^\delta(\dot{a}_\xi) \setminus R^\delta\}$, and define $\mathbb{P}^{\xi'}$ analogously.

(vi)) $\mathbb{S}^\xi := \{p \in \mathbb{Q}^{\leq\kappa} : \forall\delta \in \mathrm{dom}(E^{\leq\kappa})\, \forall\beta \in R^\delta : \mathrm{supp}(p(\delta)) \subseteq \mathrm{supp}^\delta(\dot{a}_\xi) \wedge \Delta^{p(\delta)(\beta)} \subseteq \Delta^{\delta,\beta}(\dot{a}_\xi)\}$, and define $\mathbb{S}^{\xi'}$ analogously.

These sets, as well as the actions of φ, ψ, φ' and ψ' on them, are depicted in Figure 4.1. Note that $\mathbb{R}^{\mathsf{R}} \cup \mathbb{R}^\xi \cup \mathbb{P}^\xi \subseteq \mathbb{S}^\xi$.

Fact 4.1. The following properties are very easy to verify.

(i) $\psi|_{\mathbb{R}^{\mathsf{R}}} = \psi|_{\mathbb{R}^{\mathsf{U}}} = \psi|_{\mathbb{R}^\xi} = \mathrm{id}$, and the same for ψ' in place of ψ.

(ii) $\psi[\mathbb{U}] = \mathbb{P}^\xi$, and analogously $\psi'[\mathbb{U}] = \mathbb{P}^{\xi'}$.

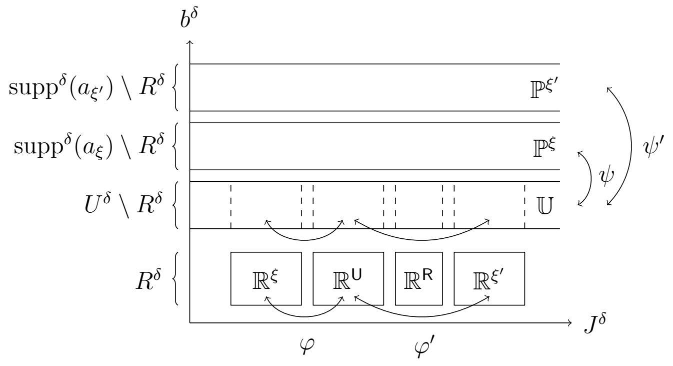

Figure 4.1: The supports at coordinate $\delta \in \mathrm{dom}(E^{\leq\kappa})$ of conditions in the sets defined in Definition 4.7, and how the maps φ, ψ, φ' and ψ' act on these sets. Note that conditions in $\mathbb{S}^\xi$ live in the union of the regions labeled $\mathbb{P}^\xi, \mathbb{R}^\xi$ and $\mathbb{R}^{\mathsf{R}}$ (and analogously for $\mathbb{S}^{\xi'}$).

(iii) $\varphi|_{\mathbb{R}^{\mathsf{R}}} = \mathrm{id}$, and the same for φ' in place of φ.

(iv) $\varphi[\mathbb{R}^{\mathsf{U}}] = \mathbb{R}^\xi$, and analogously $\varphi'[\mathbb{R}^{\mathsf{U}}] = \mathbb{R}^{\xi'}$.

(v) $\psi|_{\mathbb{S}^{\xi'}} = \mathrm{id}$, and analogously $\psi'|_{\mathbb{S}^\xi} = \mathrm{id}$.

(vi) $\varphi|_{\mathbb{S}^{\xi'}} = \mathrm{id}$, and analogously $\varphi'|_{\mathbb{S}^\xi} = \mathrm{id}$.

(vii) $\varphi|_{\mathbb{P}^\xi} = \mathrm{id}$, and analogously $\varphi'|_{\mathbb{P}^{\xi'}} = \mathrm{id}$.

Definition 4.8. Let $\delta \in \mathrm{dom}(E^{\leq\kappa})$ and let q and q' be $\mathbb{Q}^\delta$-conditions such that for all $\beta \in b^\delta : \Delta^{q(\delta)(\beta)} \cap \Delta^{q'(\delta)(\beta)} = \emptyset$. We define the condition $q + q' := \langle q(\delta)(\beta) \cup q'(\delta)(\beta) : \beta \in b^\delta \rangle$.

Furthermore, if p and p' are $\mathbb{Q}^{\leq\kappa}$ conditions such that for all $\delta \in \mathrm{dom}(E^{\leq\kappa})$ and all $\beta \in b^\delta : \Delta^{q(\delta)(\beta)} \cap \Delta^{q'(\delta)(\beta)} = \emptyset$, we define

$$p \oplus p' := \langle p(\delta) + p'(\delta) : \delta \in \mathrm{dom}(E^{\leq\kappa}) \rangle.$$

Fact 4.2. For every $\theta \in \{\psi, \varphi, \psi', \varphi'\} : \theta(p \oplus p') = \theta(p) \oplus \theta(p')$.

Recall that the nice name $\dot{z}$ is of the form $\dot{z} = \bigcup_{\alpha\in\kappa}\{\check{\alpha}\} \times A_\alpha(\dot{z})$. Let $\alpha \in \kappa$ and $q \in A_\alpha(\dot{z})$. By construction, for every $\delta \in \mathrm{dom}(E^{\leq\kappa}) : \mathrm{supp}(q(\delta)) \subseteq U^\delta$. We can therefore decompose q as $q = \bar{q} \oplus u$, where for every $\delta \in \mathrm{dom}(E^{\leq\kappa}) : \mathrm{supp}(\bar{q}(\delta)) \subseteq R^\delta$ and $\mathrm{supp}(u(\delta)) \subseteq U^\delta \setminus R^\delta$. Again by construction, we have for every $\beta \in b^\delta$:

$\Delta^{q(\delta)(\beta)} \subseteq U^{\delta,\beta}$. We can thus further decompose $\bar{q}$ as $q^{\mathsf{R}} \oplus q^{\mathsf{U}}$, where $\Delta^{q^{\mathsf{R}}(\delta)(\beta)} \subseteq R^{\delta,\beta}$ and $\Delta^{q^{\mathsf{U}}(\delta)(\beta)} \subseteq U^{\delta,\beta} \setminus R^{\delta,\beta}$.

This gives us a decomposition $q = q^{\mathsf{R}} \oplus q^{\mathsf{U}} \oplus u$, where $q^{\mathsf{R}} \in \mathbb{R}^{\mathsf{R}}$, $q^{\mathsf{U}} \in \mathbb{R}^{\mathsf{U}}$ and $u \in \mathbb{U}$.

Lemma 4.1. Define the automorphism

$$\bar{\chi} := \psi \circ \varphi \circ \varphi' \circ \psi' \circ \varphi \circ \psi,$$

i.e., we have $\chi = \psi' \circ \varphi' \circ \bar{\chi} \circ \varphi \circ \psi$. Then, $\bar{\chi}|_{A_\alpha(\dot{z})} = \mathrm{id}$ for every $\alpha \in \kappa$.

Proof. Let $\alpha \in \kappa$ and $q \in A_\alpha(\dot{z})$. We decompose $q = q^{\mathsf{R}} \oplus q^{\mathsf{U}} \oplus u$ as described above. From Fact 4.2 it follows that $\bar{\chi}(q) = \bar{\chi}(q^{\mathsf{R}}) \oplus \bar{\chi}(q^{\mathsf{U}}) \oplus \bar{\chi}(u)$, and it therefore suffices to show that q^{R}, q^{U} and u are fixed by $\bar{\chi}$. We use Fact 4.1.

Claim 4.1. $\bar{\chi}(q^{\mathsf{R}}) = q^{\mathsf{R}}$.

Proof. This is clear, since all of ψ, φ, ψ' and φ' are the identity on $\mathbb{R}^{\mathsf{R}}$, by (i). $\vdash_{\text{Claim}}$

Claim 4.2. $\bar{\chi}(q^{\mathsf{U}}) = q^{\mathsf{U}}$.

Proof. Firstly, $\psi(q^{\mathsf{U}}) = q^{\mathsf{U}}$ by (i). Next, $\varphi(q^{\mathsf{U}}) \in \mathbb{R}^{\xi}$ by (iv). Thus, $\varphi(q^{\mathsf{U}})$ is fixed by the next two automorphisms ψ' and then φ', by (i) and (vi), respectively. Then we again apply φ to get $\varphi(\varphi(q^{\mathsf{U}})) = q^{\mathsf{U}}$. Finally, q^{U} is fixed by ψ by (i). $\vdash_{\text{Claim}}$

Claim 4.3. $\bar{\chi}(u) = u$.

Proof. Firstly, $\psi(u) \in \mathbb{P}^{\xi}$ by (ii). Thus, $\psi(u)$ is fixed by φ by (vii), by ψ' by (v), by φ' by (vi) and then again by φ by (vii). The final application of ψ gives $\psi(\psi(u)) = u$. $\vdash_{\text{Claim}}$

This finishes the proof of Lemma 4.1. □

We are now ready to prove that χ does what we want it to do.

Lemma 4.2. The automorphism

$$\chi := \psi' \circ \varphi' \circ \psi \circ \varphi \circ \varphi' \circ \psi' \circ \varphi \circ \psi \circ \varphi \circ \psi$$

satisfies $\chi(\dot{a}_\xi) = \dot{a}_{\xi'}$ and $\chi(\dot{a}_{\xi'}) = \dot{a}_\xi$.

Proof. We begin with the first equality. We have $\dot{a}_\xi = \bigcup_{\alpha\in\kappa}\{\check{\alpha}\} \times A_\alpha(\dot{a}_\xi)$ and thus, $\chi(\dot{a}_\xi) = \bigcup_{\alpha\in\kappa}\{\check{\alpha}\} \times \chi[A_\alpha(\dot{a}_\xi)]$. Therefore, we must show that for every $\alpha \in \kappa$: $\chi[A_\alpha(\dot{a}_\xi)] = A_\alpha(\dot{a}_{\xi'})$.

Let $\alpha \in \kappa$. First, we deal with $\chi[A_\alpha(\dot{a}_\xi)] \subseteq A_\alpha(\dot{a}_{\xi'})$. Thus, let $p \in A_\alpha(\dot{a}_\xi)$. We know that $\varphi \circ \psi(\dot{a}_\xi) = \dot{z}$, which implies that $q := \varphi \circ \psi(p) \in A_\alpha(\dot{z})$. We also know that $\psi' \circ \varphi'(\dot{z}) = \dot{a}_{\xi'}$, and therefore $\psi' \circ \varphi'(q) \in A_\alpha(\dot{a}_{\xi'})$. Since $\bar{\chi}(q) = q$ by Lemma 4.1, we indeed obtain

$$\chi(p) = \psi' \circ \varphi' \circ \bar{\chi} \circ \varphi \circ \psi(p) \in A_\alpha(\dot{a}_{\xi'}).$$

The reverse inclusion $A_\alpha(\dot{a}_\xi) \supseteq \chi^{-1}[A_\alpha(\dot{a}_{\xi'})]$ follows from essentially the same proof: Note that $\chi^{-1} = \psi \circ \varphi \circ \bar{\chi}^{-1} \circ \varphi' \circ \psi'$, and by Lemma 4.1, $\bar{\chi}^{-1}$ is the identity on $A_\alpha(\dot{z})$ as well.

To show the second equality, i.e., $\chi(\dot{a}_{\xi'}) = \dot{a}_\xi$, we again fix $\alpha \in \kappa$ and show $\chi[A_\alpha(\dot{a}_{\xi'})] = A_\alpha(\dot{a}_\xi)$. Here, we have to deal with the entire χ at once, we again use Fact 4.1. To verify the direction "$\subseteq$", let $p' \in A_\alpha(\dot{a}_{\xi'})$. Since $p' \in \mathbb{S}^{\xi'}$, we have $\psi(p') = p'$ by (v) and $\varphi(p') = p'$ by (vi). The next two automorphisms map p' to $\varphi'(\psi'(p'))$, which is equal to a condition $q \in A_\alpha(\dot{z})$ since $\varphi'(\psi'(\dot{a}_{\xi'})) = \dot{z}$. Then, q is mapped to $\psi(\varphi(q))$, which is some $p \in A_\alpha(\dot{a}_\xi)$ because $\psi(\varphi(\dot{z})) = \dot{a}_\xi$. The last two automorphisms φ' and ψ' fix p, again by (v) and (vi).

Finally, the proof of the reverse inclusion $\chi[A_\alpha(\dot{a}_{\xi'})] \supseteq A_\alpha(\dot{a}_\xi)$ is analogous and left as an exercise to the reader. □ □

4.2 Globally Large Tower Spectra

Next, we show that arbitrarily large tower spectra at all regular cardinals simultaneously are consistent. In fact, we show that $\mathfrak{sp}(\mathfrak{t}_{\mathrm{cl}}(\kappa))$ can be arbitrarily large globally. The forcing notion we use is similar to a part of the forcing notion developed by Hechler in [23], designed to force the existence of many ω-towers.

Theorem 4.3. Let $\mathbf{V} \models \mathsf{GCH}$ and let E be an Easton function. There is a forcing extension of $\mathbf{V}$ in which

$$\forall\kappa \in \mathrm{dom}(E) : \mathfrak{sp}(\mathfrak{t}(\kappa)) = \mathfrak{sp}(\mathfrak{t}_{\mathrm{cl}}(\kappa)) = [\kappa^+, 2^\kappa], \text{ where } 2^\kappa = E(\kappa).$$

Here, $[\kappa^+, 2^\kappa]$ denotes the set of regular cardinals between κ^+ and 2^κ.

Proof. We begin by defining the relevant forcing notion.

Definition 4.9. Define for each $\kappa \in \mathrm{dom}(E)$ the set $\mathcal{I}^\kappa := \{\langle \kappa, \xi \rangle : \xi \in E(\kappa)\}$, which serves as an index set. The purpose of the entry κ is to ensure that the different $\mathcal{I}^\kappa$ are disjoint.

For each $\kappa \in \mathrm{dom}(E)$, let $\mathbb{T}^\kappa$ consist of conditions $q : \Delta^q \times \eta^q \to 2$, where $\Delta^q \in [\mathcal{I}^\kappa]^{<\kappa}$ and $\eta^q \in \kappa \setminus \{0\}$. Let $q' \leq q$ iff

(i) $q \subseteq q'$,

(ii) For all $\xi < \xi'$ with $\langle \kappa, \xi \rangle, \langle \kappa, \xi' \rangle \in \Delta^q$ and for all $\eta^q \leq \mu < \eta^{q'} : q'(\langle \kappa, \xi \rangle, \mu) = 0 \implies q'(\langle \kappa, \xi' \rangle, \mu) = 0$.

Let $\mathbb{T}$ be the Easton product of the $\mathbb{T}^\kappa$.

Lemma 4.3. Let $\kappa \in \mathrm{dom}(E)$ and decompose $\mathbb{T}$ as $\mathbb{T}^{\leq\kappa} \times \mathbb{T}^{>\kappa}$. Then, $\mathbb{T}^{>\kappa}$ is κ^+-closed and $\mathbb{T}^{\leq\kappa}$ satisfies the κ^+-c.c..

Proof. The first statement is easy to verify. To show the second statement, let A be a κ^+-sized set of $\mathbb{T}^{\leq\kappa}$-conditions. For each $p \in A$, let $S_p := \bigcup\{\Delta^{p(\delta)} \times \eta^{p(\delta)} : \delta \in \mathrm{supp}(p)\}$. Note that S_p has cardinality $<\kappa$. By the Δ-system Lemma, we obtain some $A' \subseteq A$ of cardinality κ^+ and for each $\delta \in \mathrm{dom}(E^{\leq\kappa})$ some $R^\delta \in [\mathcal{I}^\delta]^{<\delta}$ and some $r^\delta \in \delta$, such that for all these δ and all $p \neq p' \in A' : (\Delta^{p(\delta)} \times \eta^{p(\delta)}) \cap (\Delta^{p'(\delta)} \times \eta^{p'(\delta)}) = R^\delta \times r^\delta$. Note that the set $C := \{\delta : R^\delta \times r^\delta \neq \emptyset\}$ has cardinality $<\kappa$. For each $\delta \in C$, there is at most one $p \in A'$ with $\eta^{p(\delta)} \neq r^\delta$. By removing these $<\kappa$ many conditions, we can assume that no such p exist in A'.

The set $\bigcup\{R^\delta \times r^\delta : \delta \in C\}$ has cardinality $<\kappa$. By the GCH, we have $2^{<\kappa} = \kappa$, and we can therefore assume that for all $p, p' \in A'$ and all $\delta \in \mathrm{dom}(E^{\leq\kappa})$, the functions $p(\delta)$ and $p'(\delta)$ agree on the intersection of their domains. It is now easy to verify that the conditions in A' are pairwise compatible. $\square$

It follows by standard methods that

Corollary 4.2. $\mathbb{T}$ preserves cofinalities and hence cardinals.

See, for example, the proof of Easton's Theorem in [29, Ch. VIII, Lemma 4.6].

Proposition 4.1. Let $\mathbf{V} \models \mathrm{GCH}$ and let G be $\mathbb{T}$-generic over $\mathbf{V}$. Then, for any $\kappa \in \mathrm{dom}(E)$ and any regular $\lambda \in [\kappa^+, E(\kappa)]$, there is a κ-tower of height λ consisting of clubs in $\mathbf{V}[G]$.

Proof. Let $\kappa \in \mathrm{dom}(E)$ and $\lambda \in [\kappa^+, E(\kappa)]$. As before, decompose $\mathbb{T}$ as $\mathbb{T}^{\leq\kappa} \times \mathbb{T}^{>\kappa}$ and $G = G^{\leq\kappa} \times G^{>\kappa}$ accordingly. Since $\mathbb{T}^{>\kappa}$ is κ^+-closed, the GCH at $\delta \leq \kappa$ still holds in $\mathbf{V}[G^{>\kappa}]$ and $(\mathbb{T}^{\leq\kappa})^{\mathbf{V}[G^{>\kappa}]} = (\mathbb{T}^{\leq\kappa})^{\mathbf{V}}$. We work in $\mathbf{W} := \mathbf{V}[G^{>\kappa}]$.

Since κ and λ are fixed and since we are only interested in the κ-th coordinate of each $\mathbb{T}^{\leq\kappa}$-condition p, define for notational simplicity for each $p \in \mathbb{T}^{\leq\kappa}$ the following abbreviation q_p:

(i) $\forall \xi \in E(\kappa)\, \forall \alpha \in \kappa : q_p(\xi, \alpha) := p(\kappa)(\langle \kappa, \xi \rangle, \alpha)$

(ii) $\Delta^{q_p} := \Delta^{p(\kappa)}$

(iii) $\eta^{q_p} := \eta^{p(\kappa)}$.

In $\mathbf{W}[G^{\leq\kappa}]$, define for each $\xi \in E(\kappa)$ the κ-real $g_\xi := \{\alpha \in \kappa : \exists p \in G^{\leq\kappa} : q_p(\xi, \alpha) = 1\}$. We assume that $\lambda < E(\kappa)$ and define $a_\xi := \mathrm{cl}(g_\xi \setminus g_\lambda)$ for all $\xi < \lambda$. We show that the sequence $\langle a_\xi : \xi \in \lambda \rangle$ is a κ-tower of height λ in $\mathbf{W}[G^{\leq\kappa}]$. If $\lambda = E(\kappa)$, it follows by a very similar but simplified argument that setting $a_\xi := \mathrm{cl}(g_\xi)$ yields a κ-tower of height $E(\kappa)$.

It is easy to see that $\langle g_\xi : \xi \in E(\kappa) \rangle$ is well-ordered by $\supseteq^*$, and therefore, $\langle a_\xi : \xi \in \lambda \rangle$ is as well. In order to show that $\langle a_\xi : \xi \in \lambda \rangle$ does not have a pseudo-intersection in $\mathbf{W}[G^{\leq\kappa}]$, let $\dot{x}$ be a $\mathbb{T}^{\leq\kappa}$-name for a subset of κ and $p_0 \in G^{\leq\kappa}$ a condition such that $p_0 \Vdash$ "$|\dot{x}| = \kappa$". For each $\alpha \in \kappa$, let A_α be a maximal antichain deciding "$\alpha \in \dot{x}$". By the κ^+-c.c. of $\mathbb{T}^{\leq\kappa}$, the set $\Delta := \bigcup\{\Delta^{q_p} : p \in A_\alpha,\ \alpha \in \kappa\}$ has cardinality at most κ. Thus, by regularity of λ, there exists $\langle \kappa, \xi_0 \rangle \in \mathcal{I}^\kappa$ such that $\xi < \xi_0 < \lambda$ for every $\xi < \lambda$ with $\langle \kappa, \xi \rangle \in \Delta$. We show that for every $\nu \in \kappa$, the set of conditions forcing "$\dot{x} \setminus \nu \not\subseteq \dot{a}_{\xi_0}$" is dense below p_0.

Let $p \leq p_0$. By extending p, we can assume that $\langle \kappa, \lambda \rangle \in \Delta^{q_p}$. Since $p \Vdash$ "$|\dot{x}| = \kappa$", there exists $\alpha_0 > \max\{\eta^{q_p}, \nu\}$ and $\bar{p} \leq p$ with $\bar{p} \Vdash \check{\alpha}_0 \in \dot{x}$. Therefore $\bar{p}$ is compatible with some $r \in A_{\alpha_0}$ via some common extension s. In particular, p and r are compatible via s. Without loss of generality, we can assume that $\langle \xi_0, \alpha_0 \rangle, \langle \lambda, \alpha_0 \rangle \in \mathrm{dom}(q_s)$.

Note that for all $\xi_0 \leq \xi < \lambda$ and all $\max\{\eta^{q_p}, \nu\} \leq \alpha \leq \alpha_0$ with $\langle \xi, \alpha \rangle \in \mathrm{dom}(q_s)$: $\langle \xi, \alpha \rangle \notin \mathrm{dom}(q_p) \cup \mathrm{dom}(q_r)$ since $\alpha \geq \eta^{q_p}$ and by the choice of ξ_0. Therefore, we can set $\bar{s}$ equal to s except that for all such ξ and α : $q_{\bar{s}}(\xi, \alpha) := \min\{q_s(\xi, \alpha), q_s(\lambda, \alpha)\}$. It follows that $\bar{s}$ is a common extension of p and r, and for every $\max\{\eta^{q_p}, \nu\} \leq \alpha \leq \alpha_0$: $\bar{s} \Vdash$ "$\check{\alpha} \in \dot{g}_{\xi_0} \implies \check{\alpha} \in \dot{g}_\lambda$". Thus, $\bar{s} \Vdash$ "$\check{\alpha}_0 \in \dot{x} \setminus \mathrm{cl}(\dot{g}_{\xi_0} \setminus \dot{g}_\lambda)$", finishing the proof of the proposition. □

Lastly, it can be checked easily, by counting nice $\mathbb{T}^{\leq\kappa}$-names for subsets of κ, that $\forall \kappa \in \mathrm{dom}(E) : 2^\kappa = E(\kappa)$ in every $\mathbb{T}$-generic extension of $\mathbf{V} \models \mathsf{GCH}$. □

Corollary 4.3. In the above extension, $\mathfrak{b}(\kappa) = \kappa^+$ for every $\kappa \in \operatorname{dom}(E)$.

Proof. For uncountable κ, this follows from Lemma 2.10. In the case $\kappa = \omega$, it can easily be seen that the forcing notion $\mathbb{T}^\omega$ densely embeds into the part of the forcing notion introduced by Hechler [23] that deals with towers. The first author, Koelbing and Wohofsky [16, Corollary 5.1] have shown that the latter forces $\mathfrak{b}(\omega) = \omega_1$, by showing that it decomposes as a finite support iteration of Mathias forcings that preserve the unboundedness of ground model scales. □

4.3 A Locally Bounded Tower Spectrum

Our final result establishes that the κ-tower spectrum may consistently have any upper bound below 2^κ, where this upper bound is given by $\mathfrak{b}(\kappa)$.

Theorem 4.4. Assume $\mathbf{V} \models \mathsf{GCH}$. Let $\kappa < \beta$ be regular and let μ be such that $\operatorname{cf}(\mu) \geq \beta$. There is a generic extension of $\mathbf{V}$ in which

$$\mathfrak{sp}(\mathfrak{t}(\kappa)) \subseteq [\kappa^+, \mathfrak{b}(\kappa)], \text{ where } \mathfrak{b}(\kappa) = \beta \text{ and } 2^\kappa = \mu.$$

Proof. We begin by briefly sketching the idea of the proof. We force $\mathfrak{b}(\kappa) = \beta$ and $2^\kappa = \mu$ using a non-linear iteration of κ-Hechler forcing. Non-linear iterations of Hechler forcing at ω were introduced by Hechler in [24] and generalized to the uncountable by Cummings and Shelah in [11]. The strategy is to force the existence of a cofinal embedding from some partial order $\mathbb{Q}$ into the partial order $({}^\kappa\kappa, \leq^*)$, where an order-preserving embedding $f : \mathbb{Q} \to \mathbb{Q}'$ is cofinal iff $\forall p \in \mathbb{Q}' \ \exists q \in \mathbb{Q} : p \leq_{\mathbb{Q}'} f(q)$. By choosing a $\mathbb{Q}$ with appropriate bounding and dominating properties, one obtains the desired values of $\mathfrak{b}(\kappa)$ and $\mathfrak{d}(\kappa)$ in the extension. These properties are formalized by the following definition.

Definition 4.10. Let $\mathbb{Q}$ be a partially ordered set. We say that $B \subseteq \mathbb{Q}$ is unbounded iff $\forall q \in \mathbb{Q} \ \exists p \in B : p \not\leq_{\mathbb{Q}} q$. Let $\mathfrak{b}(\mathbb{Q})$ be the minimal cardinality of an unbounded subset of $\mathbb{Q}$ and let $\mathfrak{d}(\mathbb{Q})$ be the minimal cardinality of a cofinal (or dominating) subset of $\mathbb{Q}$. Thus, $\mathfrak{b}(\kappa) = \mathfrak{b}(({}^\kappa\kappa, \leq^*))$ and $\mathfrak{d}(\kappa) = \mathfrak{d}(({}^\kappa\kappa, \leq^*))$.

The following fact is easy to check.

Fact 4.3. If $f : \mathbb{Q} \to \mathbb{Q}'$ is a cofinal embedding, then $\mathfrak{b}(\mathbb{Q}') = \mathfrak{b}(\mathbb{Q})$ and $\mathfrak{d}(\mathbb{Q}') = \mathfrak{d}(\mathbb{Q})$.

Therefore, by choosing a $\mathbb{Q}$ satisfying $\mathfrak{b}(\mathbb{Q}) = \beta$ and $\mathfrak{d}(\mathbb{Q}) = \mu$ in the forcing extension, we will obtain $\mathfrak{b}(\kappa) = \beta$ and $2^\kappa \geq \mathfrak{d}(\kappa) = \mu$. The reverse inequality $2^\kappa \leq \mu$ will follow by counting nice names.

We then show that there are no κ-towers of height greater than β in the forcing extension, again due to an isomorphism of names. For this argument to succeed, we first use a preparatory forcing to obtain a particular partial order, one in which every element only lies above few others. This complication stems from the fact that we need to iterate along a well-founded partial order, where $\mathbb{Q}$ is well-founded if every $C \subseteq \mathbb{Q}$ contains a minimal element. While it is folklore that every partial order contains a cofinal, well-founded subset, choosing any such subset in our proof will not yield the upper bound we aim for. Note however that the preparatory forcing step could be skipped if we were to start with an inaccessible β.

Lemma 4.4. Assume β is regular, $\beta^{<\beta} = \beta$ and μ is such that $\mathrm{cf}(\mu) \geq \beta$. Consider the partial order $([\mu]^{<\beta}, \subseteq)$. There is a β-closed, β^+-c.c. forcing notion $\mathbb{P}$ that adds a cofinal subset $\mathbb{Q}^* \subseteq [\mu]^{<\beta}$, satisfying

(i) $\mathbb{Q}^*$ is well-founded,

(ii) For all $x \in \mathbb{Q}^* : |\{y \in \mathbb{Q}^* : y \subseteq x\}| < \beta$,

(iii) $\mathfrak{b}(\mathbb{Q}^*) = \beta$,

(iv) $\mathfrak{d}(\mathbb{Q}^*) = |\mathbb{Q}^*| = \mu$.

Proof. Let p be a $\mathbb{P}$-condition iff p is a well-founded subset of $[\mu]^{<\beta}$ of cardinality $<\beta$. The order is given by

$$q \leq p :\iff p \subseteq q \text{ and } \forall x \in p\, \forall y \in q \setminus p : y \nsubseteq x.$$

Claim 4.4. $\mathbb{P}$ is β-closed and satisfies the β^+-c.c..

Proof. Checking the first part is routine. For the second part, let $A \in [\mathbb{P}]^{\beta^+}$. Applying the Δ-system Lemma to the family $\{\bigcup p : p \in A\}$ yields some $A' \subseteq A$ of cardinality β^+ and a root $R \in [\mu]^{<\beta}$. There are at most $2^{<\beta} = \beta$ many subsets of R, and since $\beta^{<\beta} = \beta$, we can assume that $p \cap \mathcal{P}(R)$ does not depend on $p \in A'$. It follows that the $p \in A'$ are pairwise compatible. $\vdash_{\text{Claim}}$

Now, let H be $\mathbb{P}$-generic over $\mathbf{V}$ and define $\mathbb{Q}^* := \bigcup H$. By the above claim, cardinalities and cofinalities are preserved in $\mathbf{V}[H]$ and we have $([\mu]^{<\beta})^{\mathbf{V}[H]} = ([\mu]^{<\beta})^{\mathbf{V}}$.

It is easy to see that for every $x \in [\mu]^{<\beta}$, the set $\mathcal{D}_x := \{p \in \mathbb{P} : \exists y \in p : y \supseteq x\}$ is open dense in $\mathbb{P}$, by adding $\bigcup p \cup x$ to the p in question. Thus, $\mathbb{Q}^*$ is indeed cofinal in $[\mu]^{<\beta}$. Well-foundedness of $\mathbb{Q}^*$ follows from H being directed. By the same reason, we have that for every $x \in \mathbb{Q}^* : \{y \in \mathbb{Q}^* : y \subseteq x\} \subseteq p$, where $p \in H$ is any condition containing x. Thus $|\{y \in \mathbb{Q}^* : y \subseteq x\}| < \beta$.

It remains to show (iii) and (iv). In order to verify $\mathfrak{b}(\mathbb{Q}^*) = \beta$ and $\mathfrak{d}(\mathbb{Q}^*) = \mu$, it suffices, by Fact 4.3, to verify $\mathfrak{b}(([\mu]^{<\beta}, \subseteq)) = \beta$ and $\mathfrak{d}(([\mu]^{<\beta}, \subseteq)) = \mu$ in $\mathbf{V}[H]$. To check the first statement, note that by regularity of β, every $B \subseteq [\mu]^{<\beta}$ of cardinality $<\beta$ is bounded. On the other hand, for any $X \in [\mu]^\beta$, the set $\{\{\eta\} : \eta \in X\}$ is unbounded, which yields $\mathfrak{b}(([\mu]^{<\beta}, \subseteq)) = \beta$.

Similarly, any $D \subseteq [\mu]^{<\beta}$ of cardinality $<\mu$ cannot be dominating since $\bigcup D \neq \mu$. This gives us $\mathfrak{d}([\mu]^{<\beta}) \geq \mu$. The reverse inequality holds because $|[\mu]^{<\beta}| = \mu$, which follows by the assumption $\mathrm{cf}(\mu) \geq \beta$ and by the GCH in $\mathbf{V}$. Since $\mathbb{Q}^*$ is itself cofinal, this also yields $|\mathbb{Q}^*| = \mu$. □

We now fix some $\mathbb{P}$-generic H and designate $\mathbf{W} := \mathbf{V}[H]$ as the new ground model. Note that since $\mathbb{P}$ is β-closed, the GCH still holds at all cardinals below β and $\rho^\kappa = \rho$ for all ρ with $\mathrm{cf}(\rho) > \kappa$.

Definition 4.11 (see [11], Theorem 1)**.** Let $\mathbb{Q}$ be any well-founded partially ordered set. Extend $\mathbb{Q}$ to $\mathbb{Q} \cup \{\mathrm{top}\}$, where $\forall a \in \mathbb{Q} : \mathrm{top} > a$. Denote by $\mathbb{Q}_a$ the partial order $\mathbb{Q}_a := \{b \in \mathbb{Q} : b < a\}$, so that $\mathbb{Q} = \mathbb{Q}_{\mathrm{top}}$. By induction, we define for each $a \in \mathbb{Q} \cup \{\mathrm{top}\}$ the forcing notion $\mathbb{D}(\mathbb{Q}_a)$. Assume $\mathbb{D}(\mathbb{Q}_b)$ is already defined for all $b < a$. We let p be a $\mathbb{D}(\mathbb{Q}_a)$-condition iff

(i) p is a function with $\mathrm{dom}(p) \in [\mathbb{Q}_a]^{<\kappa}$.

(ii) For each $b \in \mathrm{dom}(p) : p(b) = \langle s, \dot{f} \rangle$, where $s \in {}^{<\kappa}\kappa$ and $\dot{f}$ is a nice $\mathbb{D}(\mathbb{Q}_b)$-name for an element of ${}^\kappa\kappa$. That is, $\dot{f}$ is of the form $\dot{f} = \bigcup_{\langle \alpha_1, \alpha_2 \rangle \in \kappa \times \kappa} \{\mathrm{op}(\check{\alpha}_1, \check{\alpha}_2)\} \times A_{\langle \alpha_1, \alpha_2 \rangle}$, where $A_{\langle \alpha_1, \alpha_2 \rangle}$ is an antichain in $\mathbb{D}(\mathbb{Q}_b)$ and $\Vdash_{\mathbb{D}(\mathbb{Q}_b)} \dot{f} \in ({}^\kappa\check{\kappa})$.

Let $q \leq p$ iff

(a) $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$,

(b) For all $b \in \mathrm{dom}(p)$, if $p(b) = \langle s, \dot{f} \rangle$ and $q(b) = \langle t, \dot{g} \rangle$, then $s \subseteq t$ and

$$q|_{\mathbb{Q}_b} \Vdash_{\mathbb{D}(\mathbb{Q}_b)} \begin{cases} \forall \eta \in \kappa : \dot{f}(\eta) \leq \dot{g}(\eta) \ \text{ and} \\ \forall \eta \in \mathrm{dom}(t) \setminus \mathrm{dom}(s) : t(\eta) > \dot{f}(\eta). \end{cases}$$

Finally, $\mathbb{D}(\mathbb{Q}) = \mathbb{D}(\mathbb{Q}_{\mathrm{top}})$.

Lemma 4.5. Let $\mathbb{Q}$ be any well-founded partial order. Then the following holds.

(i) $\mathbb{D}(\mathbb{Q})$ is κ-closed.

(ii) $\mathbb{D}(\mathbb{Q})$ satisfies the κ^+-c.c..

(iii) Let $\mathbb{A} \subseteq \mathbb{Q}$ be downward-closed, i.e., for all $p \in \mathbb{A}$ and $q \in \mathbb{Q} : q \leq_{\mathbb{Q}} p \implies q \in \mathbb{A}$. Then $\mathbb{D}(\mathbb{A})$ is a complete suborder of $\mathbb{D}(\mathbb{Q})$.

(iv) Assume $|\mathbb{Q}|^\kappa = |\mathbb{Q}|$. There are at most $|\mathbb{Q}|$ many nice $\mathbb{D}(\mathbb{Q})$-names for subsets of κ.

Proof. Parts (i) and (ii) are proved in [11, Claims 1 and 2]. Part (iii) is straightforward to check. For part (iv), let $|\mathbb{Q}| = \rho$ with $\rho^\kappa = \rho$ and let $a \in \mathbb{Q} \cup \{\text{top}\}$. Assume by induction that for all $b < a$ there are at most ρ many nice $\mathbb{D}(\mathbb{Q}_b)$-names for subsets of κ. In particular, there are at most ρ many nice $\mathbb{D}(\mathbb{Q}_b)$-names for elements of ${}^\kappa\kappa$. Since $\mathbb{D}(\mathbb{Q}_a)$ satisfies the κ^+-c.c., the number of nice $\mathbb{D}(\mathbb{Q}_a)$-names for subsets of κ is bounded by $|\mathbb{D}(\mathbb{Q}_a)|^\kappa$. Note that $|\mathbb{D}(\mathbb{Q}_a)| \leq |\mathbb{Q}_a|^{<\kappa} \cdot \kappa^{<\kappa} \cdot \rho^{<\kappa}$ by the induction hypothesis. This is at most ρ because $\mathbb{Q}_a \subseteq \mathbb{Q}$ and $\rho^{<\kappa} = \rho$, which finally yields that there are at most $\rho^\kappa = \rho$ nice $\mathbb{D}(\mathbb{Q}_a)$-names for subsets of κ. □

Lemma 4.6 ([11], Theorem 1)**.** Let $\mathbb{Q}$ be any well-founded partial order with $\mathfrak{b}(\mathbb{Q}) \geq \kappa^+$. In any $\mathbb{D}(\mathbb{Q})$-generic extension, $\mathbb{Q}$ can be cofinally embedded into $({}^\kappa\kappa, \leq^*)$.

Corollary 4.4. Let G be $\mathbb{D}(\mathbb{Q}^*)$-generic over $\mathbf{W}$, where $\mathbb{Q}^*$ is from Lemma 4.4. Then

$$\mathbf{W}[G] \models \mathfrak{b}(\kappa) = \beta \text{ and } 2^\kappa = \mathfrak{d}(\kappa) = \mu.$$

Proof. We have $|\mathbb{Q}^*| = \mu$ by Lemma 4.4 (iv), which implies by Lemma 4.5 (iv) that there are at most μ many nice $\mathbb{D}(\mathbb{Q}^*)$-names for subsets of κ. Thus, $\mathbf{W}[G] \models 2^\kappa \leq \mu$. In order to verify the remaining claims, it suffices by the above Lemma 4.6 and by Fact 4.3 to check that $\mathfrak{b}(\mathbb{Q}^*) = \beta$ and $\mathfrak{d}(\mathbb{Q}^*) = \mu$ still holds in $\mathbf{W}[G]$. However, this very easily follows from $\mathbb{D}(\mathbb{Q}^*)$ satisfying the κ^+-c.c.. □

Proposition 4.2. Let G be $\mathbb{D}(\mathbb{Q}^*)$-generic over $\mathbf{W}$. Then $\mathbf{W}[G] \models \mathfrak{sp}(\mathfrak{t}(\kappa)) \subseteq [\kappa^+, \beta]$.

Proof. Assume towards a contradiction that $\langle a_\xi : \xi \in \beta^+ \rangle$ is a strictly $\supseteq^*$-descending sequence in $\mathbf{W}[G]$. For each $\xi \in \beta^+$, let $\dot{a}_\xi = \bigcup_{\alpha \in \kappa} \{\check{\alpha}\} \times A^\xi_\alpha$ be a nice $\mathbb{D}(\mathbb{Q}^*)$-name for a_ξ. Assume $p_0 \in \mathbb{D}(\mathbb{Q}^*)$ is such that for all $\xi < \xi' < \beta^+ : p_0 \Vdash_{\mathbb{D}(\mathbb{Q}^*)} \dot{a}_\xi \supsetneq^* \dot{a}_{\xi'}$.

Define for every $\xi \in \beta^+$ the set

$$d_\xi := \bigcup \{x : x \in \text{dom}(p),\ p \in A^\xi_\alpha,\ \alpha \in \kappa\} \ \cup\ \bigcup \{x : x \in \text{dom}(p_0)\},$$

which is a subset of μ of size $<\beta$. Since $\mathbb{Q}^*$ is cofinal in $[\mu]^{<\beta}$, we find for each $\xi \in \beta^+$ some $D_\xi \supseteq d_\xi$ in $\mathbb{Q}^*$. As noted before, the GCH holds in $\mathbf{W}$ below β and we may therefore apply the Δ-system Lemma to the family $\{D_\xi : \xi \in \beta^+\}$ to obtain some $X \subseteq \beta^+$ of cardinality β^+ and a root R. Set $\mathbb{Q}^*_\xi := \{y \in \mathbb{Q}^* : y \subseteq D_\xi\}$ and $\mathbb{R} := \{y \in \mathbb{Q}^* : y \subseteq R\}$. Note that $\mathbb{R}$ is the root of the $\mathbb{Q}^*_\xi$. By Lemma 4.4 (ii), we have $|\mathbb{Q}^*_\xi| < \beta$, and we may therefore assume by the pigeonhole principle that $\forall \xi \in X : |\mathbb{Q}^*_\xi| = \theta < \beta$.

Claim 4.5. There exists $X' \subseteq X$ of cardinality β^+ such that for all $\xi, \xi' \in X'$, there is an order-preserving isomorphism $\psi_{\xi,\xi'} : \mathbb{Q}^*_\xi \to \mathbb{Q}^*_{\xi'}$ with $\psi_{\xi,\xi'}|_{\mathbb{R}} = \mathrm{id}$.

Proof. To see this, let L be some set of cardinality $|\mathbb{Q}^*_\xi \setminus \mathbb{R}|$ disjoint from $\mathbb{R}$. For each $\xi \in X$, we can map $\mathbb{Q}^*_\xi$ bijectively to $L \cup \mathbb{R}$, such that this bijection restricted to $\mathbb{R}$ is the identity. This bijection induces a partial order on $L \cup \mathbb{R}$. Since there are at most $2^\theta \leq \beta$ many partial orders on $L \cup \mathbb{R}$, we find the desired X' as well as the isomorphisms $\psi_{\xi,\xi'}$ by the pigeonhole principle. $\vdash_{\text{Claim}}$

Define the downward-closed partially ordered set $\mathbb{A} := \bigcup_{\xi \in X'} \mathbb{Q}^*_\xi$. Note that by definition of D_ξ, $\dot{a}_\xi$ is a nice $\mathbb{D}(\mathbb{Q}^*_\xi)$-name and thus a nice $\mathbb{D}(\mathbb{A})$-name. Furthermore, p_0 is a $\mathbb{D}(\mathbb{R})$-condition. For a fixed $\xi_0 \in X'$, the isomorphism ψ_{ξ,ξ_0} extends to an automorphism of order 2 of $\mathbb{A}$, which we denote by ψ_ξ. This automorphism ψ_ξ naturally induces an automorphism φ_ξ of $\mathbb{D}(\mathbb{A})$ in the obvious way: Let $a \in \mathbb{A} \cup \{\mathrm{top}\}$ and assume by induction that for every $b < a$, the isomorphism

$$\varphi_\xi|_{\mathbb{D}(\mathbb{A}_b)} : \mathbb{D}(\mathbb{A}_b) \to \mathbb{D}(\mathbb{A}_{\psi_\xi(b)})$$

has been defined (note the abuse of notation). In particular, this isomorphism extends to $\mathbb{D}(\mathbb{A}_b)$-names. Now let p be any $\mathbb{D}(\mathbb{A}_a)$-condition. We write for every $b \in \mathrm{dom}(p) : p(b) = \langle s(b), \dot{f}(b)\rangle$, and define

$$\varphi_\xi|_{\mathbb{D}(\mathbb{A}_a)}(p) := q, \text{ where } \begin{cases} \mathrm{dom}(q) := \psi_\xi[\mathrm{dom}(p)] \text{ and} \\ \forall \psi_\xi(b) \in \mathrm{dom}(q) : q(\psi_\xi(b)) := \langle s(b), \varphi_\xi|_{\mathbb{D}(\mathbb{A}_b)}(\dot{f})\rangle \end{cases}$$

It follows by induction that φ_ξ is an automorphism and that $\varphi_\xi|_{\mathbb{D}(\mathbb{R})} = \mathrm{id}$.

Note that $\varphi_\xi(\dot{a}_\xi)$ is a nice $\mathbb{D}(\mathbb{Q}^*_{\xi_0})$-name and that by Lemma 4.5 (iv), there are at most $|\mathbb{Q}^*_{\xi_0}| < \beta$ many nice $\mathbb{D}(\mathbb{Q}^*_{\xi_0})$-names for subsets of κ. Thus, we can extract $X'' \subseteq X'$ of cardinality β^+ such that $\varphi_\xi(\dot{a}_\xi)$ is the same nice $\mathbb{D}(\mathbb{Q}^*_{\xi_0})$-name for all $\xi \in X''$.

Fix $\xi < \xi' \in X'' \setminus \{\xi_0\}$ and define the automorphism $\chi_{\xi,\xi'} := \varphi_{\xi'} \circ \varphi_\xi \circ \varphi_{\xi'}$ of $\mathbb{A}$. By construction, $\chi_{\xi,\xi'}(\dot{a}_\xi) = \dot{a}_{\xi'}$, $\chi_{\xi,\xi'}(\dot{a}_{\xi'}) = \dot{a}_\xi$ and $\chi_{\xi,\xi'}(p_0) = p_0$. Since $\mathbb{D}(\mathbb{A})$ is a complete suborder of $\mathbb{D}(\mathbb{Q}^*)$ by Lemma 4.5 (iii), we have $p_0 \Vdash_{\mathbb{D}(\mathbb{A})} \dot{a}_\xi \supsetneq^* \dot{a}_{\xi'}$, which yields the contradiction $p_0 \Vdash_{\mathbb{D}(\mathbb{A})} \dot{a}_{\xi'} \supsetneq^* \dot{a}_\xi \wedge \dot{a}_\xi \supsetneq^* \dot{a}_{\xi'}$, just as in the proof of Theorem 4.1. □ □

Together with Lemma 2.10, the above Theorem yields the following corollary.

Corollary 4.5. Let $\kappa < \beta$ be regular uncountable and let μ be such that $\mathrm{cf}(\mu) \geq \beta$. Then, consistently,

$$\mathfrak{sp}(\mathfrak{t}_{\mathrm{cl}}(\kappa)) = \{\beta\} \text{ and } 2^\kappa = \mu.$$

As a final remark, note that by Lemma 2.10 and Lemma 2.11, the upper bound given by Theorem 4.4 is tight, in the sense that there always exists a κ-tower of height $\mathfrak{b}(\kappa)$, if κ is uncountable or if $\mathfrak{b}(\omega) < \mathfrak{d}(\omega)$. If both $\kappa = \omega$ and $\beta = \mu$ however, a well-founded cofinal subset of the partial order $([\beta]^{<\beta}, \subseteq)$ as in Lemma 4.4 is given by the well-ordered set β, in which case we have a simple finite support, β-stage linear iteration of Hechler forcing, and thus no ω-tower of height $\beta = \mathfrak{b}(\omega)$ in the extension, by Theorem 2.3.

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